

ON ONE-POINT METRIZABLE EXTENSIONS OF LOCALLY COMPACT METRIZABLE SPACES

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ABSTRACT. For a non-compact metrizable space X , let $\mathcal{E}(X)$ be the set of all one-point metrizable extensions of X , and when X is locally compact, let $\mathcal{E}_K(X)$ denote the set of all locally compact elements of $\mathcal{E}(X)$ and $\lambda : \mathcal{E}(X) \rightarrow \mathcal{Z}(\beta X \setminus X)$ be the order-anti-isomorphism (onto its image) defined in: [HJW] M. Henriksen, L. Janos and R.G. Woods, Properties of one-point completions of a non-compact metrizable space, Comment. Math. Univ. Carolinae 46 (2005), 105-123. By definition $\lambda(Y) = \bigcap_{n < \omega} \text{cl}_{\beta X}(U_n \cap X) \setminus X$, where $Y = X \cup \{p\} \in \mathcal{E}(X)$ and $\{U_n\}_{n < \omega}$ is an open base at p in Y . Answering the question of [HJW], we characterize the elements of the image of λ as exactly those non-empty zero-sets of βX which miss X , and the elements of the image of $\mathcal{E}_K(X)$ under λ , as those which are moreover clopen in $\beta X \setminus X$. We then study the relation between $\mathcal{E}(X)$ and $\mathcal{E}_K(X)$ and their order structures, and introduce a subset $\mathcal{E}_S(X)$ of $\mathcal{E}(X)$. We conclude with some theorems on the cardinality of the sets $\mathcal{E}(X)$ and $\mathcal{E}_K(X)$, and some open questions.

1. INTRODUCTION

If a Tychonoff space Y contains a space X as a dense subspace, then Y is called an *extension* of X . Two extensions Y_1 and Y_2 of X are said to be *equivalent* if there exists a homeomorphism of Y_1 onto Y_2 which keeps X pointwise fixed. This is an equivalence relation which partitions the set of Tychonoff extensions of X into equivalence classes. We identify these equivalence classes with individuals whenever no confusion arises. For two Tychonoff extensions Y_1 and Y_2 of the space X , we let $Y_1 \leq Y_2$ if there exists a continuous function from Y_2 into Y_1 which keeps X pointwise fixed. This in fact, defines a partial order on the set of all Tychonoff extensions of the space X . We refer the reader to Section 4.1 of [14] for a detailed discussion on this subject.

In this paper we are only concerned with those extensions Y of a space X for which $Y \setminus X$ is a singleton. Such kind of extensions are called *one-point extensions*. One-point extensions are studied extensively. For some results as well as some bibliographies on the subject see [11] and [12]. The present work is based on [9], in which the authors studied the one-point extensions of locally compact metrizable spaces. Their work was in turn, motivated by Bel'nov's studies of the set of all metric extensions of metrizable spaces. In [9], for a locally compact separable metrizable space X , the authors investigated the relation between the order structure of the set of all one-point metrizable extensions of X and the topology of the space $\beta X \setminus X$ (as usual βX is the Stone-Ćech compactification of X). One of

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the earliest results of this sort is due to Magill [13] who proved that if $\mathcal{K}(X)$ and $\mathcal{K}(Y)$ denote the set of all compactifications of locally compact spaces X and Y , respectively, then $\mathcal{K}(X)$ and $\mathcal{K}(Y)$ are order-isomorphic if and only if $\beta X \setminus X$ and $\beta Y \setminus Y$ are homeomorphic. Part of this paper is devoted to the relation between the order structure of certain subsets of one-point metrizable extensions of a locally compact non-separable metrizable space X and the topology of certain subspaces of $\beta X \setminus X$. We now review some of the notations and give a brief list of the main results of [9] that will be used in the sequel. This at the same time makes our work self-contained.

The letters \mathbf{I} and \mathbf{R} denote the closed unit real interval and the real line, respectively. For spaces X and Y , $C(X, Y)$ denotes the set of all continuous functions from X to Y . For $f \in C(X, \mathbf{R})$, we denote by $Z(f)$ and $\text{Coz}(f)$, the zero-set and the cozero-set of f , respectively. The support of f is the set $\text{cl}_X(\text{Coz}(f))$ and is denoted by $\text{supp}(f)$. By $\mathcal{Z}(X)$ we mean the set of all zero-sets of $f \in C(X, \mathbf{R})$. A subset of a space X is called *clopen in X* , if it is simultaneously closed and open in X . We denote by $\mathcal{B}(X)$ the set of all clopen subsets of a space X . The weight of X is denoted by $w(X)$. The letters ω and ω_1 denote the first infinite countable and the first uncountable ordinal numbers. We denote by \aleph_0 and \aleph_1 the cardinalities of ω and ω_1 , respectively. The symbol [CH] denotes the Continuum Hypothesis, and whenever it appears at the beginning of the statement of a theorem, it indicates that the Continuum Hypothesis is being assumed in that theorem. The two symbols \bigvee and \bigwedge are used to denote the least upper bound and the greatest lower bound, respectively. If P and Q are partially ordered sets, a function $f : P \rightarrow Q$ is called an *order-homomorphism* (*order-anti-homomorphism*, respectively) if $f(a) \leq f(b)$ ($f(a) \geq f(b)$, respectively) whenever $a \leq b$. The function f is called an *order-isomorphism* (*order-anti-isomorphism*, respectively) if it is moreover bijective and $f^{-1} : Q \rightarrow P$ is also an order-homomorphism (order-anti-homomorphism, respectively). The partially ordered sets P and Q are called *order-isomorphic* (*order-anti-isomorphic*, respectively) if there is an order-isomorphism (order-anti-isomorphism, respectively) between them.

Let X be a non-compact metrizable space. We denote by $\mathcal{E}(X)$ the set of all one-point metrizable extensions of X . A sequence $\mathcal{U} = \{U_n\}_{n < \omega}$ of non-empty open subsets of X is called a *regular sequence of open sets in X* , if for each $n < \omega$, $\text{cl}_X U_{n+1} \subseteq U_n$. If moreover $\bigcap_{n < \omega} U_n = \emptyset$, we call \mathcal{U} an *extension trace in X* (Definition 3.1 of [9]). Every extension trace $\{U_n\}_{n < \omega}$ in X generates a one-point metrizable extension of X . In fact if we let $Y = X \cup \{p\}$, where $p \notin X$, and define

$$\mathcal{O}_Y = \mathcal{O}_X \cup \{V \cup \{p\} : V \text{ is open in } X \text{ and } V \supseteq U_n, \text{ for some } n < \omega\}$$

where \mathcal{O}_X is the set of open subsets of X , then (Y, \mathcal{O}_Y) constitutes a one-point metrizable extension of X (see Theorem 2 of [1] or Theorem 4.3 of [2]). Conversely, if for $Y = X \cup \{p\} \in \mathcal{E}(X)$, we let for each $n < \omega$, $U_n = B(p, 1/n) \cap X$, then $\{U_n\}_{n < \omega}$ is an extension trace in X which generates Y . If $Y_{\mathcal{U}} = X \cup \{p\}$ is the one-point metrizable extension generated by the extension trace $\mathcal{U} = \{U_n\}_{n < \omega}$, then the set $\{U_n \cup \{p\}\}_{n < \omega}$ forms an open base at p in $Y_{\mathcal{U}}$. For two extension traces $\mathcal{U} = \{U_n\}_{n < \omega}$ and $\mathcal{V} = \{V_n\}_{n < \omega}$ in X , we say that \mathcal{U} is *finer than \mathcal{V}* , if for each $n < \omega$, there exists a $k_n < \omega$ such that $U_{k_n} \subseteq V_n$. For extension traces \mathcal{U} and \mathcal{V} in X , $Y_{\mathcal{U}} \geq Y_{\mathcal{V}}$ if and only if \mathcal{U} is finer than \mathcal{V} (Theorem 3.5 of [9]). For $A \subseteq X$, A^* is $(\text{cl}_{\beta X} A) \setminus X$, in particular $X^* = \beta X \setminus X$. If X is moreover locally compact, we let $\lambda : \mathcal{E}(X) \rightarrow \mathcal{Z}(X^*)$ be defined by $\lambda(Y) = \bigcap_{n < \omega} U_n^*$, where $\{U_n\}_{n < \omega}$ is an extension

trace in X which generates Y . The function λ is well-defined, and it is an order-anti-isomorphism onto its image (Theorem 4.10 of [9]). If X is moreover separable, then $\lambda(\mathcal{E}(X)) = \mathcal{Z}(X^*) \setminus \{\emptyset\}$, and therefore in this case, $\mathcal{E}(X)$ and $\mathcal{Z}(X^*) \setminus \{\emptyset\}$ are order-anti-isomorphic. Thus for locally compact non-compact separable metrizable spaces X and Y , $\mathcal{E}(X)$ and $\mathcal{E}(Y)$ are order-isomorphic if and only if X^* and Y^* are homeomorphic (Theorem 5.4 of [9]). If $\mathcal{E}_K(X)$ is the locally compact elements of $\mathcal{E}(X)$, then when X is separable, $\lambda(\mathcal{E}_K(X))$ is the set of all non-empty clopen sets of X^* (Theorems 5.5 and 5.6 of [9]). Thus when X and Y are locally compact non-compact separable metrizable spaces whose Stone-Ćech remainders are zero-dimensional, then $\mathcal{E}_K(X)$ and $\mathcal{E}_K(Y)$ are order-isomorphic if and only if X^* and Y^* are homeomorphic (Theorem 5.7 of [9]). If X is non-separable, we let σX denote the subset of βX consisting of those points of βX which are in the closure in βX of some σ -compact subset of X . We make use of the following theorem in a number of occasions (see 4.4.F of [7]).

Theorem 1.1 (Alexandroff). *If X is a locally compact non-separable metrizable space then it can be written as*

$$X = \bigoplus_{i \in I} X_i, \text{ where each } X_i \text{ is a non-compact separable subspace.}$$

Now using the above notations, for a locally compact non-separable metrizable space X , we have

$$\sigma X = \bigcup \left\{ \text{cl}_{\beta X} \left(\bigcup_{i \in J} X_i \right) : J \subseteq I \text{ is countable} \right\}.$$

Clearly σX is an open subset of βX . If $c\sigma X = \beta X \setminus \sigma X$, for a locally compact non-separable metrizable space X , if $Z \in \lambda(\mathcal{E}(X))$, then $\text{int}_{c\sigma X}(Z \setminus \sigma X) = \emptyset$ (Theorem 6.6 of [9]). When X is an uncountable discrete space the converse also holds, i.e., for $\emptyset \neq Z \in \mathcal{Z}(X^*)$, $Z \in \lambda(\mathcal{E}(X))$ if and only if $\text{int}_{c\sigma X}(Z \setminus \sigma X) = \emptyset$ (Theorem 6.8 of [9]). When X is locally compact, to every $\emptyset \neq Z \in \mathcal{Z}(X^*)$, there corresponds a regular sequence of open sets $\{U_n\}_{n < \omega}$ in X for which $Z = \bigcap_{n < \omega} U_n^*$ (Corollary 4.3 of [9]). Theorem 6.7 of [9] characterizes the elements of $\lambda(\mathcal{E}(X))$ in terms of the regular sequences of open sets generating them as follows. Suppose that X is a locally compact non-separable metrizable space and let $\emptyset \neq Z \in \mathcal{Z}(X^*)$. Then $Z \in \lambda(\mathcal{E}(X))$ if and only if there does not exist $S \in \mathcal{Z}(X)$ such that $\emptyset \neq S^* \setminus \sigma X \subseteq Z \setminus \sigma X$ if and only if $\text{cl}_{\beta X}(\bigcap_{n < \omega} U_n) \subseteq \sigma X$, where $\{U_n\}_{n < \omega}$ is a regular sequence of open sets in X for which $Z = \bigcap_{n < \omega} U_n^*$, if and only if $\bigcap_{n < \omega} U_n$ is σ -compact (with $\{U_n\}_{n < \omega}$ as in the previous condition).

We also make use of the following well known result. If Z_1, \dots, Z_n are zero-sets in X , then $\text{cl}_{\beta X}(\bigcap_{i=1}^n Z_i) = \bigcap_{i=1}^n \text{cl}_{\beta X} Z_i$. For other undefined terms and notations we refer the reader to the texts of [7], [8] and [14].

2. CHARACTERIZATION OF THE IMAGE OF λ

In Theorem 6.8 of [9], the authors characterized the image of λ for an uncountable discrete space X , as the set of all non-empty zero-sets of X^* such that $\text{int}_{c\sigma X}(Z \setminus \sigma X) = \emptyset$. They also asked whether this characterization can be generalized to the case when X is any locally compact non-separable metrizable space. In the following theorem we answer this question by characterizing those spaces X for which the above characterization of the image of λ holds.

Theorem 2.1. *Let X be a locally compact non-separable metrizable space and let $X = \bigoplus_{i \in I} X_i$, where each X_i is a separable non-compact subspace. Then we have*

- (1) *If at most countably many of the X_i 's are non-discrete, then for a $\emptyset \neq Z \in \mathcal{Z}(X^*)$, we have $Z \in \lambda(\mathcal{E}(X))$ if and only if $\text{int}_{c\sigma X}(Z \setminus \sigma X) = \emptyset$;*
- (2) *If uncountably many of the X_i 's are non-discrete, then there exists a $\emptyset \neq Z \in \mathcal{Z}(X^*)$ such that $\text{int}_{c\sigma X}(Z \setminus \sigma X) = \emptyset$, but $Z \notin \lambda(\mathcal{E}(X))$.*

Proof. 1) This follows by a modification of the proof given in Theorem 6.8 of [9].

2) Let $L = \{i \in I : X_i \text{ is not a discrete space}\}$. Suppose that $\{L_n\}_{n < \omega}$ is a partition of L into mutually disjoint uncountable subsets. For convenience, let the metric on X be chosen to be bounded by 1, and such that $d(x, y) = 1$, if x and y do not belong to the same factor X_i . Since for each $i \in L$, X_i is non-discrete, there exists a non-trivial convergent sequence $\{x_n^i\}_{n < \omega}$ in X_i . Let, for each $i \in L$, x_i denote the limit of the sequence $\{x_n^i\}_{n < \omega}$ and assume that $x_i \notin \{x_n^i\}_{n < \omega}$. We define a sequence $\{U_n\}_{n < \omega}$ by letting

$$U_n = \bigcup \left\{ B_d\left(x_i, \frac{1}{n+k}\right) : k < \omega \text{ and } i \in L_k \right\}.$$

Clearly $\{U_n\}_{n < \omega}$ is a regular sequence of open sets in X . By Lemma 4.1 of [9], we know that $Z = \bigcap_{n < \omega} U_n^* \in \mathcal{Z}(X^*)$. We claim that $\text{int}_{c\sigma X}(Z \setminus \sigma X) = \emptyset$. So suppose to the contrary that $\text{int}_{c\sigma X}(Z \setminus \sigma X) \neq \emptyset$, and let U be an open set in βX such that

$$\emptyset \neq U \setminus \sigma X \subseteq \text{cl}_{\beta X} U \setminus \sigma X \subseteq Z \setminus \sigma X.$$

For each $n < \omega$, since $\text{cl}_X U_{n+1} \subseteq U_n$, $\text{cl}_X U_{n+1}$ and $X \setminus U_n$ are disjoint zero-sets of X and thus we have $\text{cl}_{\beta X}(\text{cl}_X U_{n+1}) \cap \text{cl}_{\beta X}(X \setminus U_n) = \emptyset$. Therefore $\text{cl}_{\beta X}(\text{cl}_X U_{n+1}) \subseteq \beta X \setminus \text{cl}_{\beta X}(X \setminus U_n)$. On the other hand, since $\beta X \setminus \text{cl}_{\beta X}(X \setminus U_n) \subseteq \text{cl}_{\beta X} U_n$, we have $\beta X \setminus \text{cl}_{\beta X}(X \setminus U_n) \subseteq \text{int}_{\beta X}(\text{cl}_{\beta X} U_n)$, and therefore $\text{cl}_{\beta X} U_{n+1} \subseteq \text{int}_{\beta X}(\text{cl}_{\beta X} U_n)$. But since

$$\text{cl}_{\beta X} U \setminus \sigma X \subseteq Z \setminus \sigma X \subseteq \text{cl}_{\beta X} U_{n+1} \setminus \sigma X$$

it follows that

$$\text{cl}_{\beta X} U \setminus \text{int}_{\beta X}(\text{cl}_{\beta X} U_n) \subseteq \text{cl}_{\beta X} U \setminus \text{cl}_{\beta X} U_{n+1} \subseteq \sigma X.$$

Therefore by compactness, for each $n < \omega$, there exists a countable set $J_n \subseteq I$ such that

$$\text{cl}_{\beta X} U \setminus \text{int}_{\beta X}(\text{cl}_{\beta X} U_n) \subseteq \text{cl}_{\beta X} \left(\bigcup_{i \in J_n} X_i \right) \subseteq \text{cl}_{\beta X} \left(\bigcup_{i \in J} X_i \right)$$

where $J = J_1 \cup J_2 \cup \dots$. Now comparing with above, for each $n < \omega$, we have

$$\text{cl}_{\beta X} U \subseteq \text{cl}_{\beta X} U_n \cup \text{cl}_{\beta X} \left(\bigcup_{i \in J} X_i \right)$$

and thus

$$\text{cl}_{\beta X} U \subseteq \left(\bigcap_{n < \omega} \text{cl}_{\beta X} U_n \right) \cup \text{cl}_{\beta X} \left(\bigcup_{i \in J} X_i \right).$$

Therefore

$$U \cap X \subseteq \text{cl}_{\beta X} U \cap X \subseteq \bigcup_{i \in J} X_i \cup \left(\bigcap_{n < \omega} U_n \right) = \bigcup_{i \in J} X_i \cup \{x_i : i \in L\}.$$

Let

$$M = \bigcup_{i \in L} (\{x_i\} \cup \{x_n^i : n < \omega\}).$$

Then as $\{x_i : i \in L\} \subseteq M$ we have

$$\text{cl}_{\beta X} U = \text{cl}_{\beta X}(U \cap X) \subseteq \text{cl}_{\beta X}\left(\bigcup_{i \in J} X_i\right) \cup \text{cl}_{\beta X}(\{x_i : i \in L\}) \subseteq \sigma X \cup \text{cl}_{\beta X} M$$

and so $\text{cl}_{\beta X} U \setminus \sigma X \subseteq \text{cl}_{\beta X} M \setminus \sigma X$. For each $n < \omega$, let $V_n = U_n \cap M$. Then since for each $n < \omega$, $\{x_i : i \in L\} \subseteq V_n$, it follows that $U \cap X \subseteq V_n \cup \bigcup_{i \in J} X_i$, and therefore

$$U \cap X \subseteq \text{cl}_{\beta X} U \subseteq \text{cl}_{\beta X} V_n \cup \text{cl}_{\beta X}\left(\bigcup_{i \in J} X_i\right) \subseteq \text{cl}_{\beta X} V_n \cup \sigma X.$$

Thus by the way we have chosen U , for each $n < \omega$, we have

$$\emptyset \neq U \setminus \sigma X \subseteq \text{cl}_{\beta X} U \setminus \sigma X \subseteq \text{cl}_{\beta X} V_n \setminus \sigma X \subseteq \text{cl}_{\beta X} M \setminus \sigma X.$$

Now since M is a closed subset of the (normal) space X , by Corollary 3.6.8 of [7], $\text{cl}_{\beta X} M$ is a compactification of M equivalent to βM . But M is zero-dimensional and hence strongly zero-dimensional, as it is a locally compact metrizable space, (see Theorem 6.2.10 of [7]) therefore, there exists a non-empty clopen subset W of $\text{cl}_{\beta X} M$ such that $W \subseteq U \cap \text{cl}_{\beta X} M$ and $W \setminus \sigma X \neq \emptyset$. Since

$$W \setminus \sigma X \subseteq U \setminus \sigma X \subseteq \text{cl}_{\beta X} V_n \setminus \sigma X$$

and by the way we defined V_n (V_n is a clopen subset of M) $\text{cl}_{\beta X} V_n$ is a clopen subset of $\text{cl}_{\beta X} M$, $W \setminus \text{cl}_{\beta X} V_n$ is a compact subset of σX . Therefore, for each $n < \omega$, there exists a countable $H_n \subseteq I$ such that

$$W \setminus \text{cl}_{\beta X} V_n \subseteq \text{cl}_{\beta X}\left(\bigcup_{i \in H_n} X_i\right) \subseteq \text{cl}_{\beta X}\left(\bigcup_{i \in H} X_i\right)$$

where $H = H_1 \cup H_2 \cup \dots$. We have

$$W \subseteq (W \setminus \text{cl}_{\beta X} V_n) \cup \text{cl}_{\beta X} V_n \subseteq \text{cl}_{\beta X}\left(\bigcup_{i \in H} X_i\right) \cup \text{cl}_{\beta X} V_n$$

and therefore

$$W \subseteq \text{cl}_{\beta X}\left(\bigcup_{i \in H} X_i\right) \cup \left(\bigcap_{n < \omega} \text{cl}_{\beta X} V_n\right).$$

Now

$$W \cap M \subseteq \left(\left(\bigcup_{i \in H} X_i\right) \cap M\right) \cup \left(\bigcap_{n < \omega} V_n\right).$$

We also have

$$\bigcap_{n < \omega} V_n = \bigcap_{n < \omega} U_n \cap M = \{x_i : i \in L\}$$

and therefore $W \cap M \subseteq P \cup \{x_i : i \in L\}$, where P is a countable subset of M . For each $i \in L$ for which $x_i \in W \cap M$, since W is an open subset of $\text{cl}_{\beta X} M$, $W \cap M$ is an open neighborhood of x_i in M , and therefore as $\{x_n^i\}_{n < \omega}$ converges to x_i , there exists an $n_i < \omega$ such that $x_{n_i}^i \in W \cap M$, which implies $x_{n_i}^i \in P$. Now since P is countable, the set $Q = \{i \in L : x_i \in W \cap M\}$ is also countable and we have

$$W \cap M \subseteq P \cup \{x_i : i \in Q\} \subseteq \bigcup_{i \in G} X_i$$

for some countable subset G of I . But W is chosen to be clopen in $\text{cl}_{\beta X} M$, therefore

$$W = \text{cl}_{\beta X} W \cap \text{cl}_{\beta X} M = \text{cl}_{\beta X}(W \cap M) \subseteq \text{cl}_{\beta X}\left(\bigcup_{i \in G} X_i\right) \subseteq \sigma X$$

which is a contradiction, since W is chosen such that $W \setminus \sigma X \neq \emptyset$. This shows that $\text{int}_{\sigma X}(Z \setminus \sigma X) = \emptyset$. Now we note that by Theorem 6.7 of [9], $Z \in \lambda(\mathcal{E}(X))$ implies that

$$\text{cl}_{\beta X}(\{x_i : i \in L\}) = \text{cl}_{\beta X}\left(\bigcap_{n < \omega} U_n\right) \subseteq \sigma X$$

and so

$$\text{cl}_{\beta X}(\{x_i : i \in L\}) \subseteq \text{cl}_{\beta X}\left(\bigcup_{i \in F} X_i\right)$$

for some countable $F \subseteq I$. Therefore since $\bigcup_{i \in F} X_i$ is clopen in X ,

$$\{x_i : i \in L\} \subseteq \bigcup_{i \in F} X_i$$

which is clearly a contradiction, since we are assuming that L is uncountable. This shows that $Z \notin \lambda(\mathcal{E}(X))$, which completes the proof. \square

In the next theorem we give a characterization of the image of λ . Note that if X is locally compact, then X^* is closed in βX and thus it is C -embedded.

Theorem 2.2. *Let X be a locally compact non-compact metrizable space. Then $\lambda(\mathcal{E}(X))$ consists of exactly those non-empty zero-sets of βX which miss X .*

Proof. Suppose that $S \in \lambda(\mathcal{E}(X))$. Then $S \in \mathcal{Z}(X^*)$, and therefore there exists an $f \in C(\beta X, \mathbf{I})$ such that $Z(f) \setminus X = S$. Let for each $n < \omega$, $U_n = X \cap f^{-1}([0, 1/n))$. Then as in the proof of Lemma 4.2 of [9], $\{U_n\}_{n < \omega}$ is a regular sequence of open sets in X such that $S = \bigcap_{n < \omega} U_n^*$. Now since $Z(f) \cap X = \bigcap_{n < \omega} U_n$, it follows from Theorem 6.7 of [9] that $Z(f) \cap X$ is σ -compact. Let $Z(f) \cap X = \bigcup_{n < \omega} K_n$, where each K_n is compact, and let for each $n < \omega$, $g_n \in C(\beta X, \mathbf{I})$ be such that $g_n(K_n) \subseteq \{1\}$ and $g_n(Z(f) \setminus X) \subseteq \{0\}$. Let $g = \sum g_n/2^n$. Then g is continuous and $S = Z(f) \cap Z(g)$ is a zero-set in βX which misses X .

Conversely, suppose that $\emptyset \neq S \in \mathcal{Z}(\beta X)$ is such that $S \cap X = \emptyset$. Let $S = Z(f)$, for some $f \in C(\beta X, \mathbf{I})$. For each $n < \omega$, we let $U_n = X \cap f^{-1}([0, 1/n))$. Then by the proof of Lemma 4.2 of [9], $\{U_n\}_{n < \omega}$ is a regular sequence of open sets in X such that $S = \bigcap_{n < \omega} U_n^*$. Now since $\bigcap_{n < \omega} U_n = \emptyset$, $\{U_n\}_{n < \omega}$ is an extension trace in X , and thus $S \in \lambda(\mathcal{E}(X))$. \square

3. ON THE ORDER STRUCTURE OF THE SETS $\mathcal{E}_K(X)$ AND $\mathcal{E}(X)$ AND THEIR RELATIONSHIP

In Theorems 5.5 and 5.6 of [9], for a locally compact separable non-compact metrizable space X , the authors characterized the image of $\mathcal{E}_K(X)$ under λ as the set of all non-empty clopen subsets of X^* . We will extend this result to the non-separable case in bellow. First in the following lemma we characterize the elements of $\mathcal{E}_K(X)$ in terms of the extension traces generating them.

Lemma 3.1. *Let X be a locally compact non-compact metrizable space and let $Y = X \cup \{p\} \in \mathcal{E}(X)$. Then the following conditions are equivalent.*

- (1) *Y is locally compact;*
- (2) *For every extension trace $\mathcal{U} = \{U_n\}_{n < \omega}$ in X generating Y , there exists a $k < \omega$ such that for all $n \geq k$, $\text{cl}_X U_n \setminus U_{n+1}$ is compact;*
- (3) *There exists an extension trace $\mathcal{U} = \{U_n\}_{n < \omega}$ in X generating Y , such that for all $n < \omega$, $\text{cl}_X U_n \setminus U_{n+1}$ is compact.*

Proof. (1) implies (2). Let $\mathcal{U} = \{U_n\}_{n < \omega}$ be an extension trace in X which generates Y . Since $\{U_n \cup \{p\}\}_{n < \omega}$ forms an open base at p in Y , there exists a $k < \omega$ such that $\text{cl}_Y(U_k \cup \{p\})$ is compact. Now for each $n \geq k$, $\text{cl}_X U_n \setminus U_{n+1}$ is a closed subset of $\text{cl}_Y(U_k \cup \{p\})$, and therefore is compact.

That (2) implies (3) is trivial. (3) implies (1). Let $\mathcal{U} = \{U_n\}_{n < \omega}$ be an extension trace in X which generates Y , and suppose that $\text{cl}_X U_n \setminus U_{n+1}$ is compact for all $n < \omega$. Let $W = U_1 \cup \{p\}$, and suppose that $\{V_i\}_{i \in I}$ is an open cover of $\text{cl}_Y W$ in Y . Let $j \in I$ be such that $p \in V_j$, and let $m < \omega$ be such that $U_m \cup \{p\} \subseteq V_j$. Now since each of $\text{cl}_X U_n \setminus U_{n+1}$ is compact, a finite subset of $\{V_i\}_{i \in I}$ covers $\text{cl}_Y W$. \square

Theorem 3.2. *Let X be a locally compact non-compact metrizable space. Then $\lambda(\mathcal{E}_K(X))$ consists of exactly those elements of $\lambda(\mathcal{E}(X))$ which are clopen in X^* .*

Proof. We assume that X is non-separable. Assume the notations of Theorem 1.1. Suppose that $\mathcal{U} = \{U_n\}_{n < \omega}$ is an extension trace in X which generates $Y \in \mathcal{E}_K(X)$. By Lemma 3.1, we may assume that $\text{cl}_X U_n \setminus U_{n+1}$ is compact for all $n < \omega$. Since $\bigcap_{n < \omega} U_n = \emptyset$, we have $\text{cl}_X U_1 = \bigcup_{n < \omega} (\text{cl}_X U_n \setminus U_{n+1})$. But for each $n < \omega$, $\{X_i\}_{i \in I}$ is an open cover of $\text{cl}_X U_n \setminus U_{n+1}$, and therefore there exist finite subsets $J_n \subseteq I$ such that $\text{cl}_X U_n \setminus U_{n+1} \subseteq \bigcup_{i \in J_n} X_i$. Let $J = J_1 \cup J_2 \cup \dots$, and let $M = \bigcup_{i \in J} X_i$. Then clearly \mathcal{U} is also an extension trace in M , for which by the above lemma, the corresponding one-point metrizable extension of M is locally compact. Now since M is separable, by Theorem 5.5 of [9], $P = \bigcap_{n < \omega} \text{cl}_{\beta M} U_n \setminus M$ is a clopen subset of $\text{cl}_{\beta X} M \setminus M$, which is itself a clopen subset of X^* as M is clopen in X , and therefore it is a clopen subset of X^* . We note that $\lambda(Y) = P$.

Now suppose that $Z \in \lambda(\mathcal{E}(X))$ is clopen in X^* . First we note that by Lemma 6.6 of [9], we have $Z \setminus \sigma X = \text{int}_{c\sigma X}(Z \setminus \sigma X) = \emptyset$, and therefore $Z \subseteq \sigma X$. It follows from the latter that there exists a countable $J \subseteq I$ such that $Z \subseteq \text{cl}_{\beta X}(\bigcup_{i \in J} X_i)$. Let $M = \bigcup_{i \in J} X_i$. Now Z is a clopen subset of $\text{cl}_{\beta X} M \setminus M$, and since M is separable, it follows from Theorem 5.6 of [9] and Lemma 3.1 that $Z = \bigcap_{n < \omega} (\text{cl}_{\beta M} U_n \setminus M)$, for some extension trace $\mathcal{U} = \{U_n\}_{n < \omega}$ in M for which $\text{cl}_M U_n \setminus U_{n+1}$ is compact for each $n < \omega$. But \mathcal{U} is an extension trace in X , and since $\text{cl}_X U_n \setminus U_{n+1} = \text{cl}_M U_n \setminus U_{n+1}$ is compact, its corresponding one-point metrizable extension of X , denoted by Y , is locally compact. Now we note that $Z = \lambda(Y)$. \square

The following lemma is implicit in the proof of Theorem 3.2.

Lemma 3.3. *Let X be a locally compact non-separable metrizable space. Assume the notations of Theorem 1.1. Then for each countable $J \subseteq I$, we have $(\bigcup_{i \in J} X_i)^* \in \lambda(\mathcal{E}(X))$.*

Lemma 3.4. *Suppose that X is a locally compact non-compact metrizable space and let $Z \in \lambda(\mathcal{E}(X))$. If $S \in \mathcal{Z}(X^*)$ is such that $\emptyset \neq S \subseteq Z$, then $S \in \lambda(\mathcal{E}(X))$.*

Proof. Let $T \in \mathcal{Z}(\beta X)$ be such that $S = T \setminus X$. Now $S = Z \cap T$ misses X , and thus by Theorem 2.2 $S \in \lambda(\mathcal{E}(X))$. \square

The following theorem gives another characterization of the image of $\mathcal{E}_K(X)$ under λ .

Theorem 3.5. *Let X be a locally compact non-compact metrizable space. Then $\lambda(\mathcal{E}_K(X))$ consists of exactly those non-empty zero-sets of X^* which are of the form $X^* \setminus \text{cl}_{\beta X}(Z(f))$, where $f \in C(X, \mathbf{I})$ is of σ -compact support.*

Proof. Suppose that $S \in \lambda(\mathcal{E}_K(X))$. Then since $\{\text{cl}_{\beta X} Z \setminus X : Z \in \mathcal{Z}(X)\}$ forms a base for closed subsets of X^* , there exists a collection \mathcal{C} of zero-sets of X such that $S = \bigcup \{X^* \setminus \text{cl}_{\beta X} Z : Z \in \mathcal{C}\}$. Since S is compact, there exists a finite number of zero-sets Z_1, \dots, Z_n such that $S = \bigcup_{i=1}^n (X^* \setminus \text{cl}_{\beta X} Z_i) = X^* \setminus \text{cl}_{\beta X} Z$, where $Z = Z_1 \cap \dots \cap Z_n \in \mathcal{Z}(X)$. Let $Z = Z(f)$, for some $f \in C(X, \mathbf{I})$. If X is separable, then trivially $\text{supp}(f)$ is σ -compact. So suppose that X is non-separable and assume the notations of Theorem 1.1. Let $L = \{i \in I : \text{Coz}(f) \cap X_i \neq \emptyset\}$. Then there exists a zero-set $T \in \mathcal{Z}(X)$ such that $T \subseteq \text{Coz}(f)$ and for each $i \in L$, $T \cap X_i \neq \emptyset$. Now since $T \cap Z(f) = \emptyset$, we have $\text{cl}_{\beta X} T \cap \text{cl}_{\beta X} Z(f) = \emptyset$, and therefore $\text{cl}_{\beta X} T \subseteq \beta X \setminus \text{cl}_{\beta X} Z(f)$. But S is a clopen subset of X^* , and so by Lemma 6.6 of [9], $S \subseteq \sigma X$. This implies that $\text{cl}_{\beta X} T \subseteq \text{cl}_{\beta X} (\bigcup_{i \in J} X_i)$, for some countable $J \subseteq I$. It follows that $T \subseteq \bigcup_{i \in J} X_i$, and therefore L is countable. Now clearly $\text{supp}(f)$, being a closed subset of the separable space $\bigcup_{i \in J} X_i$, is σ -compact (see 3.8.C of [7]).

Conversely, suppose that $\emptyset \neq S \in \mathcal{Z}(X^*)$ is of the form $X^* \setminus \text{cl}_{\beta X}(Z(f))$, for some $f \in C(X, \mathbf{I})$ of σ -compact support. If X is separable then clearly S , being a clopen subset of X^* , is in $\lambda(\mathcal{E}_K(X))$. Suppose that X is non-separable. Then by definition of σX , since $\text{supp}(f)$ is σ -compact, we have $S \subseteq \text{cl}_{\beta X}(\text{supp}(f)) \subseteq \sigma X$, and therefore $S \subseteq \text{cl}_{\beta X}(\bigcup_{i \in J} X_i)$, for some countable $J \subseteq I$. Thus $S \in \lambda(\mathcal{E}_K(X))$. \square

In the following theorem, assuming [CH], we give a purely order-theoretic description of $\mathcal{E}_K(X)$ as a subset of $\mathcal{E}(X)$.

Theorem 3.6. [CH] *Let X be a locally compact non-separable metrizable space. For a set $\mathcal{F} \subseteq \mathcal{E}(X)$ consider the following conditions.*

- (1) *For each $A \in \mathcal{F}$, $|\{Y \in \mathcal{E}(X) : Y > A\}| \leq \aleph_1$;*
- (2) *If $A \in \mathcal{E}(X)$ is such that $|\{Y \in \mathcal{E}(X) : Y > A\}| \leq \aleph_1$, then there exists a $B \in \mathcal{F}$ such that $B < A$;*
- (3) *For each $A, B \in \mathcal{F}$ such that $A < B$, there exists a $C \in \mathcal{F}$ such that $B \wedge C = A$ and B and C have no common upper bound in $\mathcal{E}(X)$.*

Then the set $\mathcal{E}_K(X)$ is the largest (with respect to set-theoretic inclusion) subset of $\mathcal{E}(X)$ satisfying the above three conditions.

Proof. First we verify that $\mathcal{E}_K(X)$ satisfies the above conditions. To show that condition (1) is satisfied, let $A \in \mathcal{E}_K(X)$. Then we have $\lambda(A) \subseteq \sigma X$ (see Lemma 6.6 of [9]) and therefore, assuming the notations of Theorem 1.1, $\lambda(A) \subseteq \text{cl}_{\beta X} M$, where $M = \bigcup_{i \in G} X_i$, for some countable $G \subseteq I$. Now if $Y \in \mathcal{E}(X)$ is such that $Y > A$, then $\lambda(Y)$ is a zero-set in $\text{cl}_{\beta X} M$. But $|\mathcal{Z}(\text{cl}_{\beta X} M)| \leq \aleph_1$, as M is separable, and thus condition (1) holds.

Now we show that $\mathcal{E}_K(X)$ satisfies condition (2). So suppose that $A \in \mathcal{E}(X)$ is such that $|\{Y \in \mathcal{E}(X) : Y > A\}| \leq \aleph_1$. First we show that $\lambda(A) \subseteq \sigma X$. Suppose the contrary, and let $\mathcal{V} = \{V_n\}_{n < \omega}$ be an extension trace in X which generates A . For each $n < \omega$, let $H_n = \{i \in I : V_n \cap X_i \neq \emptyset\}$. Then since we are assuming that $\lambda(A) \setminus \sigma X \neq \emptyset$, each H_n is an uncountable subset of I . We consider the following two cases.

Case 1) Suppose that $\bigcap_{n < \omega} H_n$ is uncountable. Let $K \subseteq \bigcap_{n < \omega} H_n$, with $|K| = \aleph_1$. For each non-empty $L \subseteq K$ and each $n < \omega$ let

$$W_L^n = \left(\bigcup_{i \in L} X_i \right) \cap V_n.$$

Then each $\mathcal{W}_L = \{W_L^n\}_{n < \omega}$ is an extension trace in X finer than \mathcal{V} , and \mathcal{W}_{L_1} and \mathcal{W}_{L_2} are non-equivalent for distinct non-empty $L_1, L_2 \subseteq K$. But this is a contradiction, as by our assumption the number of these extension traces cannot be greater than \aleph_1 .

Case 2) Suppose that $\bigcap_{n < \omega} H_n$ is countable. We define a sequence $\{k_n\}_{n < \omega}$ of positive integers as follows. Let $k_1 = 1$. Then since $H_1 \supseteq H_2 \supseteq \dots$ and H_{k_1} is uncountable, arguing inductively there exists a sequence $k_1 < k_2 < \dots$ with $H_{k_n} \setminus H_{k_{n+1}}$ being uncountable for each $n < \omega$. We may assume that $H_n \setminus H_{n+1}$ is uncountable for each $n < \omega$. Suppose that $|K| = \aleph_1$, and let for each $n < \omega$, $K_n \subseteq H_n \setminus H_{n+1}$ be such $|K_n| = \aleph_1$. We use K as an index set to (faithfully) index the elements of K_n . Thus $K_n = \{k_i^n : i \in K\}$. For each non-empty $L \subseteq K$ and each $n < \omega$ let $L_n = \{k_i^n : i \in L\}$, and define

$$W_L^n = \bigcup \{V_n \cap X_i : i \in L_n \cup L_{n+1} \cup \dots\}.$$

Then each $\mathcal{W}_L = \{W_L^n\}_{n < \omega}$ is an extension trace in X finer than \mathcal{V} , and they are non-equivalent for distinct non-empty $L_1, L_2 \subseteq K$. But this is again a contradiction.

Therefore $\lambda(A) \subseteq \sigma X$ and we can assume that $\lambda(A) \subseteq P^*$ properly, where $P = \bigcup_{i \in H} X_i$, and $H \subseteq I$ is countable. Let $\lambda(B) = P^*$. Then $B \in \mathcal{E}_K(X)$ and $B < A$. Thus $\mathcal{E}_K(X)$ satisfies condition (2).

Next, to show that $\mathcal{E}_K(X)$ satisfies condition (3), suppose that $A, B \in \mathcal{E}_K(X)$ are such that $A < B$. Let $C \in \mathcal{E}_K(X)$ be such that $\lambda(C) = \lambda(A) \setminus \lambda(B)$. Then clearly $B \wedge C = A$ and thus condition (3) holds for $\mathcal{E}_K(X)$.

Now suppose that a set $\mathcal{F} \subseteq \mathcal{E}(X)$ satisfies conditions (1)-(3). Let $A \in \mathcal{F}$. Then by condition (1), $|\{Y \in \mathcal{E}(X) : Y > A\}| \leq \aleph_1$. Arguing as above we have $\lambda(A) \subseteq \sigma X$. Let $\lambda(A) \subseteq Q^*$, where $Q = \bigcup_{i \in J} X_i$ and $J \subseteq I$ is countable. Let $B \in \mathcal{E}(X)$ be such that $\lambda(B) = Q^*$. Then since $|\{Y \in \mathcal{E}(X) : Y > B\}| \leq \aleph_1$, using condition (2), there exists a $C \in \mathcal{F}$ such that $C < B$. Therefore $C < A$, and so by condition (3), there exists a $D \in \mathcal{F}$ such that $A \wedge D = C$ and A and D have no common upper bound in $\mathcal{E}(X)$. Therefore $\lambda(A) \cap \lambda(D) = \emptyset$. Suppose that $x \in \lambda(B) \setminus (\lambda(A) \cup \lambda(D))$. Let $f \in C(\beta X, \mathbf{I})$ be such that $f(x) = 1$ and $f(\lambda(A) \cup \lambda(D)) = \{0\}$. Let $S = Z(f) \cap \lambda(C)$. Then since $C \leq A$, $S \neq \emptyset$, and therefore $S = \lambda(E)$, for some $E \in \mathcal{E}(X)$. Clearly since $\lambda(A) \subseteq S$, we have $E \leq A$. But $\lambda(D) \subseteq Z(f)$ and $C \leq D$, therefore $\lambda(D) \subseteq S$, and thus $E \leq D$. This combined with $E \leq A$ implies that $E \leq C$. But $x \in \lambda(B) \subseteq \lambda(C)$ and $x \notin S$. This contradiction shows that $\lambda(B) \setminus \lambda(A) \subseteq \lambda(D)$. Finally, we note that by the above inclusion $\lambda(B) \setminus \lambda(D) \subseteq \lambda(A)$, and conversely, if $x \in \lambda(A)$, then since $B \leq A$, and $\lambda(A) \cap \lambda(D) = \emptyset$, we have $x \in \lambda(B) \setminus \lambda(D)$. Therefore $\lambda(A) = \lambda(B) \setminus \lambda(D)$, and thus $\lambda(A)$ is clopen in X^* . This shows that $A \in \mathcal{E}_K(X)$, and therefore $\mathcal{F} \subseteq \mathcal{E}_K(X)$, which together with the first part of the proof, establishes the theorem. \square

Theorem 3.7. *Let X be a locally compact non-compact metrizable space. Then $\mathcal{E}_K(X)$ and $\mathcal{E}(X)$ are never order-isomorphic.*

Proof. *Case 1)* Suppose that X is separable. Suppose that $\mathcal{E}_K(X)$ is order-isomorphic to $\mathcal{E}(X)$, and let $\psi : \lambda(\mathcal{E}_K(X)) \rightarrow \lambda(\mathcal{E}(X))$ denote an order-isomorphism. First we show that $\lambda(\mathcal{E}_K(X)) = \lambda(\mathcal{E}(X))$, from which it follows that every non-empty zero-set of X^* is clopen in X^* , and therefore X^* is a P -space. By Proposition 1.65 of [16] every pseudocompact P -space is finite, thus X^* is finite. By 4C of [16], the Stone-Čech remainder of a non-pseudocompact space has at least $2^{2^{\aleph_0}}$ points. Therefore

X is pseudocompact and being metrizable it is compact. But this is a contradiction. Now let $X^* \neq Z \in \lambda(\mathcal{E}(X))$. Let $A = X^* \setminus \psi^{-1}(Z)$, where $A \in \lambda(\mathcal{E}_K(X))$. If $\psi(A) \cap Z \neq \emptyset$, then there exists a $B \in \lambda(\mathcal{E}_K(X))$ such that $\psi(B) = \psi(A) \cap Z$. But such a B necessarily has non-empty intersection with one of $\psi^{-1}(Z)$ or A . Now since ψ is an order-isomorphism, it is easy to see that in either case we get a contradiction. Therefore $\psi(A) \cap Z = \emptyset$. If $\psi(A) \cup Z \neq X^*$, then there exists an $\emptyset \neq H \in \mathcal{Z}(X^*)$ with $H \cap (\psi(A) \cup Z) = \emptyset$. Let $G \in \lambda(\mathcal{E}_K(X))$ be such that $\psi(G) = H$, then again we get a contradiction, as G intersects one of $\psi^{-1}(Z)$ or A . Therefore $\psi(A) \cup Z = X^*$, and thus $Z = X^* \setminus \psi(A)$, i.e., $Z \in \lambda(\mathcal{E}_K(X))$.

Case 2) Suppose that X is non-separable. Suppose to the contrary that $\mathcal{E}_K(X)$ and $\mathcal{E}(X)$ are order-isomorphic and let $\phi : \mathcal{E}_K(X) \rightarrow \mathcal{E}(X)$ denote an order-isomorphism. Since X is non-separable, there exists a sequence $\{Y_n\}_{n < \omega}$ in $\mathcal{E}(X)$ such that $Y_1 < Y_2 < \dots$. Consider $\mathcal{F} = \{\lambda(Y_n)\}_{n < \omega}$. Then \mathcal{F} has the f.i.p., and therefore $Z = \bigcap \mathcal{F} \in \lambda(\mathcal{E}(X))$. Let $Z = \lambda(Y)$, for some $Y \in \mathcal{E}(X)$. Clearly $Y = \bigvee_{n < \omega} Y_n$. Let for each $n < \omega$, $\phi(S_n) = Y_n$ and $\phi(S) = Y$. Then $S_1 < S_2 < \dots < S$. For each $n < \omega$, let $\lambda(S_n) \setminus \lambda(S) = \lambda(T_n)$, for some $T_n \in \mathcal{E}_K(X)$. Now since the sequence $\{S_n\}_{n < \omega}$ is increasing, the sequence $\{\lambda(T_n)\}_{n < \omega}$ and equivalently the sequence $\{\phi(T_n)\}_{n < \omega}$ is also increasing, and thus $\bigvee_{n < \omega} \phi(T_n) \in \mathcal{E}(X)$. Let $T \in \mathcal{E}_K(X)$ be such that $\phi(T) = \bigvee_{n < \omega} \phi(T_n)$. Let $A \in \mathcal{E}_K(X)$ be such that $\lambda(A) = \lambda(S) \cup \lambda(T)$. Now for each $n < \omega$, $T \geq T_n$, and thus $\lambda(A) \subseteq \lambda(S) \cup \lambda(T_n) = \lambda(S_n)$, i.e., for each $n < \omega$, we have $A \geq S_n$, or equivalently, $\phi(A) \geq \phi(S_n) = Y_n$. Therefore $\phi(A) \geq Y = \phi(S)$ and $A \geq S$. Thus $T \geq S$. But $T \geq T_1$, which is a contradiction as $\lambda(T_1) \cap \lambda(S) = \emptyset$. \square

Lemma 3.8. *Let X be a locally compact non-separable metrizable space. If $\emptyset \neq Z \in \mathcal{Z}(\beta X)$ then $Z \cap \sigma X \neq \emptyset$.*

Proof. Suppose that $\{x_n\}_{n < \omega}$ is an infinite sequence in σX . Then using the notations of Theorem 1.1, there exists a countable $J \subseteq I$ such that $\{x_n\}_{n < \omega} \subseteq \text{cl}_{\beta X}(\bigcup_{i \in J} X_i)$, and therefore it has a limit point in σX . Thus σX is countably compact and therefore, pseudocompact, and $v(\sigma X) = \beta(\sigma X) = \beta X$. The result now follows as for any Tychonoff space T , any non-empty zero-set of vT intersects T (see Lemma 5.11(f) of [14]). \square

Lemma 3.9. *Let X be a locally compact non-separable metrizable space. If $\emptyset \neq Z \in \mathcal{Z}(X^*)$ then $Z \cap \sigma X \neq \emptyset$.*

Proof. Let $S \in \mathcal{Z}(\beta X)$ be such that $Z = S \setminus X$. By the above lemma $S \cap \sigma X \neq \emptyset$. Suppose that $S \cap (\sigma X \setminus X) = \emptyset$. Then $S \cap \sigma X = S \cap X$. Assume the notations of Theorem 1.1 and let $L = \{i \in I : S \cap X_i \neq \emptyset\}$. Since $S \cap (\sigma X \setminus X) = \emptyset$, it follows that L is finite. We define a function $f : \beta X \rightarrow \mathbf{I}$ such that it equals to 1 on $\text{cl}_{\beta X}(\bigcup_{i \in L} X_i)$, and it is 0 otherwise. Clearly f is continuous. Since $Z(f) \cap S \in \mathcal{Z}(\beta X)$ misses σX , by the above lemma, $Z(f) \cap S = \emptyset$. But since $\beta X \setminus \sigma X \subseteq Z(f)$, we have $Z = S \cap (\beta X \setminus \sigma X) \subseteq S \cap Z(f) = \emptyset$, which is a contradiction. Therefore $Z \cap (\sigma X \setminus X) = S \cap (\sigma X \setminus X) \neq \emptyset$. \square

Lemma 3.10. *Let X be a locally compact non-separable metrizable space and let $S, T \in \mathcal{Z}(X^*)$. If $S \cap \sigma X \subseteq T \cap \sigma X$ then $S \subseteq T$.*

Proof. Suppose that $S \setminus T \neq \emptyset$. Let $x \in S \setminus T$ and let $f \in C(\beta X, \mathbf{I})$ be such that $f(x) = 0$ and $f(T) = \{1\}$. Then $Z(f) \cap S \in \mathcal{Z}(X^*)$ is non-empty and therefore by

Lemma 3.9, $Z(f) \cap S \cap \sigma X \neq \emptyset$. But this is impossible as we have $Z(f) \cap S \cap \sigma X \subseteq Z(f) \cap T = \emptyset$. \square

Theorem 3.11. *Let X be a non-compact metrizable space. Then $\mathcal{E}(X)$ has a minimum if and only if X is locally compact and separable.*

Proof. Suppose that $Y = X \cup \{p\}$ is the minimum in $\mathcal{E}(X)$. If X is not locally compact, then there exists an $x \in X$ such that for every open neighborhood U of x in X , $\text{cl}_X U$ is not compact. Let U and W be disjoint open neighborhoods of x and p in Y , respectively. Since $\text{cl}_X U$ is not compact, there exists a discrete sequence $\{V_n\}_{n < \omega}$ of non-empty open (in $\text{cl}_X U$) subsets of X , which are faithfully indexed. Consider $\mathcal{F} = \{V_n \cap U\}_{n < \omega}$. Then \mathcal{F} is a discrete sequence of non-empty open subsets of X . For each $n < \omega$, let A_n be a non-empty open subset of X such that $\text{cl}_X A_n \subseteq V_n \cap U$. For each $n < \omega$ we form a sequence $\{B_k^n\}_{k < \omega}$ of non-empty open subset of X such that $A_n \supseteq B_1^n$, and $B_k^n \supseteq \text{cl}_X B_{k+1}^n$ for each $k < \omega$. Let $m < \omega$ be such that $B(p, 1/m) \subseteq W$, and for each $n < \omega$ define

$$C_n = B\left(p, \frac{1}{m+n}\right) \cap X \text{ and } E_n = \bigcup \{B_k^n : k \geq n\}.$$

Let for each $n < \omega$, $D_n = C_n \cup E_n$. Then since $\{B_k^n\}_{k < \omega}$ is discrete, we have $\text{cl}_X D_{n+1} \subseteq D_n$, and $\bigcap_{n < \omega} D_n = \emptyset$, i.e., $\mathcal{D} = \{D_n\}_{n < \omega}$ forms an extension trace in X . Since $\mathcal{C} = \{C_n\}_{n < \omega}$ is an extension trace in X generating Y , by Theorem 3.5 of [9], \mathcal{D} is finer than \mathcal{C} , and therefore there exists a $k < \omega$ such that $D_k \subseteq C_1$. But this is a contradiction, as $C_1 \subseteq W$ and $D_k \cap U \neq \emptyset$. Therefore X is locally compact.

Suppose that X is not separable. Since $\mathcal{E}(X)$ has a minimum, $\lambda(\mathcal{E}(X))$ has a maximum. Let S denote the maximum of $\lambda(\mathcal{E}(X))$. Assume the notations of Theorem 1.1. Now since for each countable $J \subseteq I$, $(\bigcup_{i \in J} X_i)^* \subseteq S$, we have $\sigma X \setminus X \subseteq S$, and therefore by Lemma 3.10 (with S and X^* being the zero-sets) we have $S = X^*$, which is a contradiction, as X is not σ -compact (see 1B of [16]). Thus X is locally compact and separable.

The converse is clear, as in this case, the one-point compactification of X is the minimum. \square

By replacing $\mathcal{E}(X)$ by $\mathcal{E}_K(X)$ in the last part of the above proof we obtain the following result.

Theorem 3.12. *Let X be a locally compact non-separable metrizable space. Then $\mathcal{E}_K(X)$ has no minimum.*

In the next result we show that when X is a zero-dimensional locally compact metrizable space, $\mathcal{E}_K(X)$ is a cofinal subset of $\mathcal{E}(X)$. For this purpose we need the following proposition, stated in Lemma 15.17 of [6].

Proposition 3.13. *Let X be a locally compact space, let F be a nowhere dense subset of X , and let Z be a non-empty zero-set of βX which misses X . Then we have*

$$\text{int}_{X^*}(Z \setminus \text{cl}_{\beta X} F) \neq \emptyset.$$

Theorem 3.14. *Let X be a zero-dimensional locally compact non-separable metrizable space. Then for each $Y \in \mathcal{E}(X)$, there exists an $S \in \mathcal{E}_K(X)$ such that $S \geq Y$. In other words, $\mathcal{E}_K(X)$ is a cofinal subset of $\mathcal{E}(X)$. Furthermore, there is no such*

greatest S , (in fact there are at least 2^{\aleph_0} mutually incomparable elements of $\mathcal{E}_K(X)$ greater than Y) and if Y is not locally compact, there is no such least S .

Proof. Suppose that $Y \in \mathcal{E}(X)$, and let $Z = \lambda(Y)$. By Proposition 3.13 we have $\text{int}_{X^*} Z \neq \emptyset$. Now since X is strongly zero-dimensional, (see Theorem 6.2.10 of [7]) there exists a clopen subset V of βX such that $\emptyset \neq V \setminus X \subseteq Z$. Let $S \in \mathcal{E}_K(X)$ be such that $\lambda(S) = V \cap Z = V \setminus X$. Then $S \geq Y$. Now since $V \cap X$ is non-compact, there exists a discrete family $\{U_n\}_{n < \omega}$ of non-empty open subsets of $V \cap X$. Since X is locally compact and zero-dimensional, we may assume that each U_n is compact. Let $\omega = \bigcup_{t < 2^{\aleph_0}} N_t$ be a partition of ω into almost disjoint infinite sets. Let for $t < 2^{\aleph_0}$, $A_t = \bigcup_{n \in N_t} U_n$. Then each A_t is a clopen subsets of X and $\text{cl}_{\beta X} A_s \cap \text{cl}_{\beta X} A_t \subseteq X$, for $s \neq t$. Let $S_t \in \mathcal{E}_K(X)$ be such that $\lambda(S_t) = \text{cl}_{\beta X} A_t \cap Z = A_t^* \subseteq V \setminus X$. Clearly $S_t > S$, for each $t < 2^{\aleph_0}$, and they are mutually incomparable for $s \neq t$.

Now suppose that Y is not locally compact. Then Z is not clopen in X^* and therefore $V \setminus X \neq Z$. By Proposition 3.13 we have $\text{int}_{X^*}(Z \setminus V) \neq \emptyset$. Let U be a clopen subset of βX such that $\emptyset \neq U \setminus X \subseteq Z \setminus V$. Then $(U \cup V) \setminus X \subseteq Z$ is clopen in X^* and properly contains $\lambda(S)$. \square

Theorem 3.15. *Let X be a zero-dimensional locally compact non-compact metrizable space and let $S, T \in \mathcal{E}(X)$. Then $S \geq T$ if and only if for every $Y \in \mathcal{E}_K(X)$, if $Y \geq S$ then $Y \geq T$.*

Proof. One implication is trivial. Suppose that for every $Y \in \mathcal{E}_K(X)$, $Y \geq S$ implies $Y \geq T$. If $\lambda(S) \setminus \lambda(T) \neq \emptyset$, then there exists an $A \in \mathcal{Z}(\beta X)$ such that $A \cap \lambda(S) \neq \emptyset$ and $A \cap \lambda(T) = \emptyset$. Now by Proposition 3.13, we have $\text{int}_{X^*}(A \cap \lambda(S)) \neq \emptyset$, and thus there exists a clopen subset V of βX such that $\emptyset \neq V \setminus X \subseteq A \cap \lambda(S)$. Let $Y \in \mathcal{E}_K(X)$ be such that $\lambda(Y) = V \setminus X$. Then $Y \geq S$, and therefore by our assumptions $Y \geq T$. But $\lambda(Y) \cap \lambda(T) = \emptyset$, which is a contradiction. Thus $\lambda(S) \setminus \lambda(T) = \emptyset$ and $S \geq T$. \square

Corollary 3.16. *Let X be a zero-dimensional locally compact non-compact metrizable space. Then for any $S \in \mathcal{E}(X)$ we have*

$$S = \bigwedge \{Y \in \mathcal{E}_K(X) : Y \geq S\}.$$

In the next two theorems we investigate the question of existence of greatest lower bounds and least upper bounds for arbitrary subsets of $\mathcal{E}_K(X)$ and $\mathcal{E}(X)$.

Lemma 3.17. *Let X be a zero-dimensional locally compact non-compact metrizable space and let $\emptyset \neq Z \in \mathcal{Z}(X^*)$. Then the following conditions are equivalent.*

- (1) $Z \in \lambda(\mathcal{E}(X))$;
- (2) *There exists an extension trace $\{V_n\}_{n < \omega}$ in X , consisting of clopen subsets of X , such that $Z = \bigcap_{n < \omega} V_n^*$.*

Proof. That (2) implies (1) is trivial. (1) implies (2). Let $\{U_n\}_{n < \omega}$ be an extension trace in X such that $Z = \bigcap_{n < \omega} U_n^*$. Since X is strongly zero-dimensional, (see Theorem 6.2.10 of [7]) and for each $n < \omega$, $\text{cl}_X U_{n+1}$ and $X \setminus U_n$ are completely separated in X , by Lemma 6.2.2 of [7], there exists a clopen subset V_n of X such that $\text{cl}_X U_{n+1} \subseteq V_n \subseteq U_n$. Clearly now $\{V_n\}_{n < \omega}$ forms an extension trace in X which satisfies our requirements. \square

A Boolean algebra is said to be *Cantor separable* if no strictly increasing sequence has a least upper bound (see 2.20 of [16]). Proposition 2.22 of [16] states that the Boolean algebra of clopen subsets of a totally disconnected compact space without isolated points, in which every zero-set is regular-closed, is Cantor separable. This will be used in the following theorem.

Theorem 3.18. *Let X be a locally compact non-compact metrizable space. Then the following hold.*

- (1) For $Y_1, Y_2 \in \mathcal{E}_K(X)$, Y_1, Y_2 may not have any common upper bound in $\mathcal{E}_K(X)$;
- (2) For any $Y_1, \dots, Y_n \in \mathcal{E}_K(X)$ which have a common upper bound in $\mathcal{E}_K(X)$, $\bigvee_{i=1}^n Y_i$ exists in $\mathcal{E}_K(X)$;
- (3) For a sequence $\{Y_n\}_{n < \omega}$ in $\mathcal{E}_K(X)$ which has an upper bound in $\mathcal{E}_K(X)$, $\bigvee_{n < \omega} Y_n$ may not exist in $\mathcal{E}_K(X)$. In fact, if we assume X to be moreover zero-dimensional, then for any sequence $\{Y_n\}_{n < \omega}$ in $\mathcal{E}_K(X)$ with $Y_1 < Y_2 < \dots$, $\{Y_n\}_{n < \omega}$ has an upper bound in $\mathcal{E}_K(X)$ but $\bigvee_{n < \omega} Y_n$ does not exist in $\mathcal{E}_K(X)$;
- (4) For any $Y_1, \dots, Y_n \in \mathcal{E}_K(X)$, $\bigwedge_{i=1}^n Y_i$ exists in $\mathcal{E}_K(X)$;
- (5) For any sequence $\{Y_n\}_{n < \omega}$ in $\mathcal{E}_K(X)$, $\{Y_n\}_{n < \omega}$ has a lower bound in $\mathcal{E}_K(X)$;
- (6) For a sequence $\{Y_n\}_{n < \omega}$ in $\mathcal{E}_K(X)$, $\bigwedge_{n < \omega} Y_n$ may not exist in $\mathcal{E}_K(X)$. In fact, if we assume X to be moreover zero-dimensional, then for any sequence $\{Y_n\}_{n < \omega}$ in $\mathcal{E}_K(X)$ such that $Y_1 > Y_2 > \dots$, $\bigwedge_{n < \omega} Y_n$ does not exist in $\mathcal{E}_K(X)$;
- (7) An uncountable family of elements of $\mathcal{E}_K(X)$ may not have any common lower bound in $\mathcal{E}_K(X)$. In fact, if we assume X to be moreover non-separable and zero-dimensional, then there exists a subset of $\mathcal{E}_K(X)$ of cardinality \aleph_1 , with no common lower bound in $\mathcal{E}_K(X)$.

Proof. (1), (2) and (4) are straightforward. 3) Suppose that X is zero-dimensional and let $\{Y_n\}_{n < \omega}$ be a sequence in $\mathcal{E}_K(X)$ such that $Y_1 < Y_2 < \dots$. Since the sequence $\{\lambda(Y_n)\}_{n < \omega}$ is decreasing, it has the f.i.p., and therefore $S = \bigcap_{n < \omega} \lambda(Y_n) \in \lambda(\mathcal{E}(X))$. Now Proposition 3.13 implies that $\text{int}_{X^*} S \neq \emptyset$. But since X is strongly zero-dimensional (see Theorem 6.2.10 of [7]) there exists a non-empty clopen subset U of X^* such that $U \subseteq S$. Let $A \in \mathcal{E}_K(X)$ be such that $\lambda(A) = U$. Then clearly A is an upper bound for $\{Y_n\}_{n < \omega}$ in $\mathcal{E}_K(X)$. Now suppose that $\bigvee_{n < \omega} Y_n$ exists in $\mathcal{E}_K(X)$ and let $Y = \bigvee_{n < \omega} Y_n$. Consider the family $\{\lambda(Y_n) \setminus \lambda(Y)\}_{n < \omega}$ of non-empty decreasing clopen subsets of X^* . Let $T = \bigcap_{n < \omega} \lambda(Y_n) \setminus \lambda(Y) \neq \emptyset$. Then by Proposition 3.13, we have $\text{int}_{X^*} T \neq \emptyset$. Let V be a non-empty clopen subset of X^* such that $V \subseteq T$. Let $\lambda(B) = V \cup \lambda(Y)$, for some $B \in \mathcal{E}_K(X)$. Then since for any $n < \omega$, $V \subseteq T \subseteq \lambda(Y_n)$, B is an upper bound for $\{Y_n\}_{n < \omega}$, and therefore $B \geq Y$. Thus $V \subseteq \lambda(B) \subseteq \lambda(Y)$. But this is a contradiction as $V \subseteq T \subseteq X^* \setminus \lambda(Y)$.

5) We assume that X is non-separable. Let $\{Y_n\}_{n < \omega}$ be a sequence in $\mathcal{E}_K(X)$. Then for each $n < \omega$, by Lemma 6.6 of [9], we have $\lambda(Y_n) \subseteq \sigma X \setminus X$. Assuming the notations of Theorem 1.1, it follows that there exists a countable $J \subseteq I$ such that for each $n < \omega$, $\lambda(Y_n) \subseteq M^*$, where $M = \bigcup_{i \in J} X_i$. Let $\lambda(Y) = M^*$, for some $Y \in \mathcal{E}_K(X)$. Then clearly Y is a lower bound for the sequence $\{Y_n\}_{n < \omega}$.

6) Suppose that X is zero-dimensional and let $\{Y_n\}_{n < \omega}$ be a sequence in $\mathcal{E}_K(X)$ with $Y_1 > Y_2 > \dots$. Suppose that $Y = \bigwedge_{n < \omega} Y_n$ exists in $\mathcal{E}_K(X)$. First we assume that X is separable and verify that X^* is a totally disconnected compact space without isolated points in which every zero-set is regular-closed.

Clearly X^* is totally disconnected, as it is zero-dimensional (see Theorem 6.2.10 of [7]). Since X is Lindelöf, by 3.8.C of [7], it is σ -compact. By Remark 14.17 of [6], the Stone-Čech remainder of any zero-dimensional locally compact σ -compact space has no isolated points. Therefore X^* does not have any isolated points. Finally, X being Lindelöf is realcompact. By Theorem 15.18 of [6], any zero-set of the Stone-Čech remainder of a locally compact realcompact space is regular-closed, therefore, every zero-set in X^* is regular-closed. Now since $\lambda(Y_1) \subseteq \lambda(Y_2) \subseteq \dots \subseteq \lambda(Y)$ (properly), Proposition 2.22 of [16] implies the existence of an $S \in \mathcal{B}(X^*)$ such that $\lambda(Y_n) \subseteq S \subseteq \lambda(Y)$ (properly), for each $n < \omega$. Let $A \in \mathcal{E}_K(X)$ be such that $\lambda(A) = S$. Then clearly A is a lower bound for the sequence $\{Y_n\}_{n < \omega}$ but $A > Y$. This contradiction proves our theorem in this case.

Now suppose that X is non-separable. By Lemma 6.6 of [9], we have $\lambda(Y) \subseteq \sigma X \setminus X$ and $\lambda(Y_n) \subseteq \sigma X \setminus X$, for any $n < \omega$. Assume the notations of Theorem 1.1. Then $\lambda(Y) \subseteq M^*$ and $\lambda(Y_n) \subseteq M^*$, for any $n < \omega$, where $M = \bigcup_{i \in J} X_i$ and $J \subseteq I$ is countable. Then since $\text{cl}_{\beta X} M \simeq \beta M$ and M is separable, the problem reduces to the case we considered above.

7) Let X be zero-dimensional. Assume the notations of Theorem 1.1. Let $J \subseteq I$ be such that $|J| = \aleph_1$, and let $\{J_k : k < \omega_1\}$ be a partition of J into mutually disjoint subsets with $|J_k| = \aleph_0$, for any $k < \omega_1$. For any $k < \omega_1$, let $Y_k \in \mathcal{E}_K(X)$ be such that $\lambda(Y_k) = (\bigcup_{i \in J_k} X_i)^*$. We claim that the family $\mathcal{F} = \{Y_k\}_{k < \omega_1}$ has no lower bound in $\mathcal{E}_K(X)$. Suppose the contrary, and let $Y \in \mathcal{E}_K(X)$ be a lower bound for \mathcal{F} . By Lemma 3.16, there exists an extension trace $\mathcal{U} = \{U_n\}_{n < \omega}$ in X generating Y , such that each U_n is a clopen subsets of X . By Lemma 3.1, there exists an $m < \omega$ such that $U_n \setminus U_{n+1}$ is compact, for all $n \geq m$. We may assume that $m = 1$. Let $k < \omega_1$. Then

$$\left(\bigcup_{i \in J_k} X_i \right)^* = \lambda(Y_k) \subseteq \lambda(Y) = \bigcap_{n < \omega} U_n^* \subseteq U_1^*$$

and therefore

$$\text{cl}_{\beta X} \left(\bigcup_{i \in J_k} X_i \right) \subseteq \text{cl}_{\beta X} U_1 \cup X = \text{cl}_{\beta X} U_1 \cup \bigcup_{i \in I} X_i.$$

Now since U_1 is clopen in X , $\text{cl}_{\beta X} U_1$ is clopen in βX , and therefore there exists a finite set $L_k \subseteq I$ such that

$$\text{cl}_{\beta X} \left(\bigcup_{i \in J_k} X_i \right) \subseteq \text{cl}_{\beta X} U_1 \cup \bigcup_{i \in L_k} X_i \subseteq \text{cl}_{\beta X} \left(U_1 \cup \bigcup_{i \in L_k} X_i \right).$$

But $U_1 \cup \bigcup_{i \in L_k} X_i$ is clopen in X , and thus

$$\bigcup_{i \in J_k} X_i \subseteq U_1 \cup \bigcup_{i \in L_k} X_i.$$

Let for any $k < \omega_1$, $i_k \in J_k \setminus L_k$. Then $X_{i_k} \subseteq U_1$, and therefore $\bigcup \{X_{i_k} : k < \omega_1\}$ being a closed subset of the σ -compact set $U_1 = \bigcup_{n < \omega} (U_n \setminus U_{n+1})$ is Lindelöf. But this is clearly a contradiction. This completes the proof. \square

The following is a counterpart of the above theorem, which deals with the subsets of $\mathcal{E}(X)$.

Theorem 3.19. *Let X be a locally compact non-compact metrizable space. Then the following hold.*

- (1) For $Y_1, Y_2 \in \mathcal{E}(X)$, Y_1, Y_2 may not have any common upper bound in $\mathcal{E}(X)$;
- (2) For any sequence $\{Y_n\}_{n < \omega}$ in $\mathcal{E}(X)$, if $\{Y_n\}_{n < \omega}$ has an upper bound in $\mathcal{E}(X)$, then $\bigvee_{n < \omega} Y_n$ exists in $\mathcal{E}(X)$;
- (3) For any $Y_1, \dots, Y_n \in \mathcal{E}(X)$, $\bigwedge_{i=1}^n Y_i$ exists in $\mathcal{E}(X)$;
- (4) A sequence $\{Y_n\}_{n < \omega}$ in $\mathcal{E}(X)$ may not have any lower bound in $\mathcal{E}(X)$. In fact, if we moreover assume that $w(X) \geq 2^{\aleph_0}$, then there exists a sequence $\{Y_n\}_{n < \omega}$ in $\mathcal{E}(X)$ which does not have any lower bound in $\mathcal{E}(X)$.
- (5) A sequence $\{Y_n\}_{n < \omega}$ in $\mathcal{E}(X)$ which has a lower bound in $\mathcal{E}(X)$, may not have a greatest lower bound in $\mathcal{E}(X)$.

Proof. (1)-(3) are straightforward. 4) Let Δ denote the set of all increasing (i.e., $f(n) \leq f(n+1)$, for any $n < \omega$) functions $f : \omega \rightarrow \omega$ which are not eventually constant. We first check that $|\Delta| = 2^{\aleph_0}$. To show this, let for each $g \in \{0, 1\}^\omega$ which is not eventually constant, $f_g : \omega \rightarrow \omega$ be defined by $f_g(n) = n + g(n)$, for any $n < \omega$. Then clearly since for distinct $g, h \in \{0, 1\}^\omega$, $f_g \neq f_h$, we have $|\Delta| \geq 2^{\aleph_0}$. It is clear that $|\Delta| \leq 2^{\aleph_0}$. Assume the notations of Theorem 1.1. Since $w(X) = |I| \geq 2^{\aleph_0}$, for simplicity we may assume that $I \supseteq \Delta$. For any $n, k < \omega$, let

$$U_k^n = \bigcup \{X_f : f \in \Delta \text{ and } f(k) \leq n\}$$

and let $\mathcal{U}_n = \{U_k^n\}_{k < \omega}$. We verify that \mathcal{U}_n is an extension trace in X . By the way we defined U_k^n and since f is increasing we have $U_{k+1}^n \subseteq U_k^n$. Suppose that $\bigcap_{k < \omega} U_k^n \neq \emptyset$, and let $x \in \bigcap_{k < \omega} U_k^n$. Since for any $k < \omega$, $x \in U_k^n$, by definition of U_k^n , there exists an $f_k \in \Delta$ such that $f_k(k) \leq n$ and $x \in X_{f_k}$. But since the family $\{X_i\}_{i \in I}$ is faithfully indexed and $X_i \cap X_j = \emptyset$, for distinct $i, j \in I$, we obtain that $f_1 = f_2 = \dots \equiv h$. Now for any $k < \omega$, we have $h(k) = f_k(k) \leq n$, which implies h to be eventually constant, which is a contradiction. Therefore $\bigcap_{k < \omega} U_k^n = \emptyset$ and each \mathcal{U}_n is an extension trace in X . Let for any $n < \omega$, $Y_n \in \mathcal{E}(X)$ be generated by \mathcal{U}_n . We claim that $\{Y_n\}_{n < \omega}$ has no lower bound in $\mathcal{E}(X)$. So suppose to the contrary that $Y \in \mathcal{E}(X)$ is such that $Y \leq Y_n$, for any $n < \omega$. Let $\mathcal{U} = \{U_n\}_{n < \omega}$ be an extension trace in X which generates Y . Since for any $n < \omega$, we are assuming $Y \leq Y_n$, by Theorem 3.5 of [9], \mathcal{U}_n is finer than \mathcal{U} . For any $n, i < \omega$, let $k_i^n < \omega$ be such that $U_{k_i^n}^i \subseteq U_n$. We can also assume that $k_1^n < k_2^n < k_3^n < \dots$, for any $n < \omega$. We define a function $g : \omega \rightarrow \omega$ as follows.

Let $g(i) = 1$ for $i = 1, \dots, k_{t_1}^1$, where $t_1 = 1$. Inductively assume that for $n < \omega$, $t_1 < \dots < t_n$ are defined in such a way that

$$g(i) = m, \text{ for } i = k_{t_{m-1}}^{m-1} + 1, \dots, k_{t_m}^m \text{ and } m = 1, \dots, n.$$

Now since $k_1^{n+1} < k_2^{n+1} < k_3^{n+1} < \dots$, there exists a $t < \omega$ such that $k_t^{n+1} > k_{t_n}^n$ and $t > t_n$. Let $t_{n+1} = t$ and define

$$g(i) = n + 1, \text{ for } i = k_{t_n}^n + 1, \dots, k_{t_{n+1}}^{n+1}.$$

Consider the function $g : \omega \rightarrow \omega$. Clearly $g \in \Delta$ and since for any $n < \omega$, $g(k_{t_n}^n) = n$ we have $X_g \subseteq U_{k_{t_n}^n}^n$. But since the sequence $\{t_n\}_{n < \omega}$ is increasing, $t_n \geq n$, and therefore $k_{t_n}^n \geq k_n^n$. Thus $U_{k_{t_n}^n}^n \subseteq U_{k_n^n}^n$, which combined with the fact that $U_{k_n^n}^n \subseteq U_n$ implies that $X_g \subseteq U_n$, for any $n < \omega$. But this is a contradiction, as $\bigcap_{n < \omega} U_n = \emptyset$.

5) Let X be non-separable and let $\{Y_n\}_{n < \omega}$ be a sequence in $\mathcal{E}_K(X)$ such that $Y_1 > Y_2 > \dots$. By part (5) of Theorem 3.18, $\{Y_n\}_{n < \omega}$ has a lower bound in $\mathcal{E}(X)$. Suppose that $A = \bigwedge_{n < \omega} Y_n$ exists in $\mathcal{E}(X)$. Then since for each $n < \omega$, $Y_n > A$, we have $\lambda(A) \setminus \lambda(Y_n) \neq \emptyset$, and therefore $S = \bigcap_{n < \omega} (\lambda(A) \setminus \lambda(Y_n)) \in \lambda(\mathcal{E}(X))$. By

Proposition 3.13, there exists a non-empty open subset U of X^* with $U \subseteq S$. Let $x \in U$ and let $f \in C(X^*, \mathbf{I})$ be such that $f(x) = 1$ and $f(X^* \setminus U) \subseteq \{0\}$. Let $T = \lambda(A) \cap Z(f)$. Clearly for each $n < \omega$, $\lambda(Y_n) \subseteq X^* \setminus U \subseteq Z(f)$, and therefore $\lambda(Y_n) \subseteq T$. Let $Y \in \mathcal{E}(X)$ be such that $\lambda(Y) = T$. Then since for each $n < \omega$, $Y \leq Y_n$, we have $Y \leq A$. But since $x \in U \subseteq \lambda(A)$, this implies that $x \in \lambda(Y) \subseteq Z(f)$, which is a contradiction. \square

Our final result of this section deals with the cardinalities of cofinal subsets of $\mathcal{E}(X)$.

Theorem 3.20. *Let X be a locally compact non-separable metrizable space and let $\mathcal{F} \subseteq \mathcal{E}(X)$. If for each $Y \in \mathcal{E}(X)$ there exists an $A \in \mathcal{F}$ such that $A \leq Y$, ($A \geq Y$, respectively) then \mathcal{F} is uncountable.*

Proof. Suppose that for each $Y \in \mathcal{E}(X)$ there exists an $A \in \mathcal{F}$ such that $A \leq Y$. Suppose that $\mathcal{F} = \{Y_n\}_{n < \omega}$. Let for each $n < \omega$, $S_n = \lambda(Y_n)$. Then by Lemma 6.6 of [9], $\text{int}_{c\sigma X}(S_n \setminus \sigma X) = \emptyset$. Now since $c\sigma X$ is compact, by the Baire Category Theorem we have $\text{int}_{c\sigma X}(\bigcup_{n < \omega} S_n \setminus \sigma X) = \emptyset$, and therefore $c\sigma X \setminus \bigcup_{n < \omega} S_n \neq \emptyset$. Let $x \in c\sigma X \setminus \bigcup_{n < \omega} S_n$. Then since for each $n < \omega$, $x \notin S_n$, there exists a $Z_n \in \mathcal{Z}(\beta X)$ such that $x \in Z_n$ and $Z_n \cap S_n = \emptyset$. Let $Z = \bigcap_{n < \omega} Z_n$. Then since $Z \setminus X \in \mathcal{Z}(X^*)$ is non-empty, by Lemma 3.9 we have $Z \cap (\sigma X \setminus X) \neq \emptyset$. Therefore, using the notations of Theorem 1.1, for some countable $J \subseteq I$, $T = Z \cap (\bigcup_{i \in J} X_i)^* \neq \emptyset$. Let $Y \in \mathcal{E}(X)$ be such that $\lambda(Y) = T$. By assumption, $Y \geq Y_k$, for some $k < \omega$. Therefore $S_k = \lambda(Y_k) \supseteq \lambda(Y) = T$. But $T \cap S_k = \emptyset$, which is a contradiction.

To show the second part of the theorem, let for each $i \in I$, $Y_i \in \mathcal{E}(X)$ be such that $\lambda(Y_i) = X_i^*$. Let $A_i \in \mathcal{F}$ be such that $A_i \geq Y_i$. Then clearly since for $i \neq j$, $X_i^* \cap X_j^* = \emptyset$, we have $|\mathcal{F}| \geq |\{A_i : i \in I\}| = |I| = w(X)$. \square

4. THE RELATIONSHIP BETWEEN THE ORDER STRUCTURE OF THE SET $\mathcal{E}_K(X)$ AND THE TOPOLOGY OF SUBSPACES OF $\beta X \setminus X$

In Theorem 5.7 of [9] the authors proved that for locally compact separable metrizable spaces X and Y whose Stone-Ćech remainders are zero-dimensional, $\mathcal{E}_K(X)$ and $\mathcal{E}_K(Y)$ are order-isomorphic if and only if X^* and Y^* are homeomorphic. In this section we generalize this result to the case when X and Y are not separable.

Theorem 4.1. *Let X and Y be zero-dimensional locally compact non-separable metrizable spaces and let $\omega\sigma X = \sigma X \cup \{\Omega\}$ and $\omega\sigma Y = \sigma Y \cup \{\Omega'\}$ be the one-point compactifications of σX and σY , respectively. If $\mathcal{E}_K(X)$ and $\mathcal{E}_K(Y)$ are order-isomorphic then $\omega\sigma X \setminus X$ and $\omega\sigma Y \setminus Y$ are homeomorphic.*

Proof. Let $\phi : \mathcal{E}_K(X) \rightarrow \mathcal{E}_K(Y)$ be an order-isomorphism and let $g = \lambda_Y \phi \lambda_X^{-1} : \lambda_X(\mathcal{E}_K(X)) \rightarrow \lambda_Y(\mathcal{E}_K(Y))$. We define a function $G : \mathcal{B}(\omega\sigma X \setminus X) \rightarrow \mathcal{B}(\omega\sigma Y \setminus Y)$ between the two Boolean algebras of clopen sets, and verify that it is an order-isomorphism.

Set $G(\emptyset) = \emptyset$ and $G(\omega\sigma X \setminus X) = \omega\sigma Y \setminus Y$. Let $U \in \mathcal{B}(\omega\sigma X \setminus X)$. If $U \neq \emptyset$ and $\Omega \notin U$, then U is an open subset of $\sigma X \setminus X$, and therefore an open subset of X^* . Assuming the notations of Theorem 1.1, there exists a countable $J \subseteq I$ such that $U \subseteq (\bigcup_{i \in J} X_i)^*$, and thus $U \in \lambda_X(\mathcal{E}_K(X))$. In this case we let $G(U) = g(U)$. If $U \neq \omega\sigma X \setminus X$ and $\Omega \in U$, then $(\omega\sigma X \setminus X) \setminus U \in \lambda_X(\mathcal{E}_K(X))$ and we let $G(U) = (\omega\sigma Y \setminus Y) \setminus g((\omega\sigma X \setminus X) \setminus U)$.

To show that G is an order-homomorphism, let $U, V \in \mathcal{B}(\omega\sigma X \setminus X)$ with $U \subseteq V$. We may assume that $U \neq \emptyset$ and $V \neq \omega\sigma X \setminus X$. We consider the following three cases.

Case 1) If $\Omega \notin V$, then clearly $G(U) = g(U) \subseteq g(V) = G(V)$.

Case 2) Suppose that $\Omega \notin U$ and $\Omega \in V$. If $G(U) \setminus G(V) \neq \emptyset$ then $T = g(U) \cap g((\omega\sigma X \setminus X) \setminus V) \neq \emptyset$ and therefore $T \in \lambda_Y(\mathcal{E}_K(Y))$. Let $S \in \lambda_X(\mathcal{E}_K(X))$ be such that $g(S) = T$. Then since g is an order-isomorphism, we have $S \subseteq U \cap ((\omega\sigma X \setminus X) \setminus V) = \emptyset$, which is a contradiction. Therefore $G(U) \subseteq G(V)$.

Case 3) If $\Omega \in U$, then since $(\omega\sigma X \setminus X) \setminus V \subseteq (\omega\sigma X \setminus X) \setminus U$ we have

$$G(U) = (\omega\sigma Y \setminus Y) \setminus g((\omega\sigma X \setminus X) \setminus U) \subseteq (\omega\sigma Y \setminus Y) \setminus g((\omega\sigma X \setminus X) \setminus V) = G(V).$$

This shows that G is an order-homomorphism.

To complete the proof we note that since $\phi^{-1} : \mathcal{E}_K(Y) \rightarrow \mathcal{E}_K(X)$ is also an order-isomorphism, if we denote $h = \lambda_X \phi^{-1} \lambda_Y^{-1}$, then arguing as above, h induces an order-homomorphism $H : \mathcal{B}(\omega\sigma Y \setminus Y) \rightarrow \mathcal{B}(\omega\sigma X \setminus X)$ which is easy to see that $H = G^{-1}$.

To see that $\omega\sigma X \setminus X$ is zero-dimensional, we note that since X is zero-dimensional locally compact metrizable, it is strongly zero-dimensional (see Theorem 6.2.10 of [7]) i.e., βX is zero-dimensional. Thus σX is also zero-dimensional. But the one-point compactification of a locally compact non-compact zero-dimensional space is again zero-dimensional, therefore $\omega\sigma X$ and thus $\omega\sigma X \setminus X$ is zero-dimensional. Similarly $\omega\sigma Y \setminus Y$ is also zero-dimensional, and thus, they are homeomorphic by Stone Duality. \square

The following provides a converse to the above theorem under some weight restrictions. Note that here we are not assuming X and Y to be necessarily zero-dimensional.

Theorem 4.2. *Let X and Y be locally compact non-separable metrizable spaces. Suppose moreover, that at least one of X and Y has weight greater than 2^{\aleph_0} . Then if $\omega\sigma X \setminus X$ and $\omega\sigma Y \setminus Y$ are homeomorphic, $\mathcal{E}_K(X)$ and $\mathcal{E}_K(Y)$ are order-isomorphic.*

Proof. Without any loss of generality we may assume that $w(X) > 2^{\aleph_0}$. Suppose that $f : (\sigma X \setminus X) \cup \{\Omega\} \rightarrow (\sigma Y \setminus Y) \cup \{\Omega'\}$ is a homeomorphism. First we show that $f(\Omega) = \Omega'$.

Suppose that $f(\Omega) = p$, where $p \in \sigma Y \setminus Y$. Suppose that K is a countable subset of J such that $p \in \text{cl}_{\beta Y}(\bigcup_{i \in K} Y_i)$, where $Y = \bigoplus_{i \in J} Y_i$, with each Y_i being a separable non-compact subspace. Then $V = (\bigcup_{i \in K} Y_i)^*$ is an open neighborhood of p in $(\sigma Y \setminus Y) \cup \{\Omega'\}$, and therefore, there exists a neighborhood W of Ω in $(\sigma X \setminus X) \cup \{\Omega\}$ such that $f(W) \subseteq V$. Clearly we may choose W to be of the form

$$W = ((\sigma X \setminus X) \cup \{\Omega\}) \setminus \text{cl}_{\beta X} \left(\bigcup_{i \in L} X_i \right)$$

for some countable $L \subseteq I$ (with the notations of Theorem 1.1). Let $M = \bigcup_{i \in K} Y_i$. Then since M is separable we have $w(\text{cl}_{\beta Y} M) \leq 2^{\aleph_0}$. Now $\{X_i^* : i \in I \setminus L\}$ is a collection of non-empty mutually disjoint open subsets of W , and thus $w(W) \geq |I| = w(X) > 2^{\aleph_0}$. On the other hand $w(W) = w(f(W)) \leq w(\text{cl}_{\beta Y} M) \leq 2^{\aleph_0}$. This contradiction shows that $f(\Omega) = \Omega'$. Therefore $\sigma X \setminus X$ is homeomorphic to $\sigma Y \setminus Y$.

Now suppose that $Z \in \lambda_X(\mathcal{E}_K(X))$. Then by Lemma 6.6 of [9], we have $Z \subseteq \sigma X \setminus X$, and thus there exists a countable set $A \subseteq I$ such that $Z \subseteq \text{cl}_{\beta X} P$, where

$P = \bigcup_{i \in A} X_i$. But since P^* is clopen in $\sigma X \setminus X$, $f(P^*)$ is clopen in $\sigma Y \setminus Y$, and since it is also compact, there exists a countable set $B \subseteq J$ such that $f(P^*) \subseteq Q^*$, where $Q = \bigcup_{i \in B} Y_i$. But since Z is clopen in X^* , $f(Z)$ is clopen in $\sigma Y \setminus Y$, and as $f(Z) \subseteq Q^*$, it is also clopen in Q^* , and thus clopen in Y^* , i.e., $f(Z) \in \lambda_Y(\mathcal{E}_K(Y))$. Now we define a function $F : \lambda_X(\mathcal{E}_K(X)) \rightarrow \lambda_Y(\mathcal{E}_K(Y))$ by $F(Z) = f(Z)$. The function F is clearly well-defined and it is an order-homomorphism. Since f^{-1} is also a homeomorphism which takes Ω' to Ω , arguing as above, we can define a function $G : \lambda_Y(\mathcal{E}_K(Y)) \rightarrow \lambda_X(\mathcal{E}_K(X))$ by $G(Z) = f^{-1}(Z)$, which is clearly the inverse of F . Thus F is an order-isomorphism. \square

A compact zero-dimensional F -space of weight 2^{\aleph_0} in which every non-empty G_δ -set has infinite interior, is called a *Parovičenko space*. It is well known that under [CH] ω^* is the only Parovičenko space (Parovičenko Theorem, see Corollary 1.2.4 of [15]).

The following theorem is proved (assuming [CH]) in [5], for the case when X is a discrete space of cardinality \aleph_1 .

Theorem 4.3. [CH] *Let X be a zero-dimensional locally compact metrizable space of weight \aleph_1 . Then we have*

$$(\sigma X \setminus X) \cup \{\Omega\} \simeq \omega^*.$$

Proof. We verify the assumptions of the Parovičenko Theorem. Let $Y = (\sigma X \setminus X) \cup \{\Omega\}$. To show that Y is an F -space, let A and B be disjoint cozero-sets in Y . Suppose first that Ω belong to one of A and B . Without any loss of generality we may assume that $\Omega \in A$. Then since $Y \setminus A \in \mathcal{Z}(Y)$, using the notations of Theorem 1.1, we have $Y \setminus A \subseteq M^*$, where $M = \bigcup_{i \in J} X_i$, and $J \subseteq I$ is countable. Now M is σ -compact (see 3.8.C of [7]) and $\beta M \setminus M \simeq M^*$ is an F -space, (see 1.62 of [16]) thus $A \cap M^*$ and B , being disjoint cozero-sets in M^* , are completely separated in M^* . But M^* itself is clopen in Y , thus A and B are completely separated in Y . Suppose that Ω does not belong to any of A and B . Then since A and B are cozero-sets in Y , they are σ -compact, and therefore as $A, B \subseteq \sigma X \setminus X$, they are cozero-sets in P^* , where $P = \bigcup_{i \in K} X_i$, for some countable $K \subseteq I$. But as above P^* is an F -space, and A, B being completely separated in P^* , are completely separated in Y .

Next we show that Y is zero-dimensional of weight \aleph_1 . For each countable $L \subseteq I$, let $Q_L = \bigcup_{i \in L} X_i$. Then since Q_L is separable, $\text{cl}_{\beta X} Q_L$ has weight at most \aleph_1 . Now since Q_L is strongly zero-dimensional, (see Theorem 6.2.10 of [7]) by Theorem 1.1.15 of [7], we can choose a base \mathcal{C}_L consisting of clopen subsets of Q_L^* (and therefore clopen in Y) such that $|\mathcal{C}_L| \leq \aleph_1$. Let

$$\mathcal{D} = \bigcup \{ \mathcal{C}_L : L \subseteq I \text{ is countable} \} \cup \{ Y \setminus \text{cl}_{\beta X} Q_L : L \subseteq I \text{ is countable} \}.$$

Then clearly \mathcal{D} forms a base consisting of clopen subsets of Y , and therefore Y is zero-dimensional of weight $w(Y) \leq \aleph_1$. But $\{X_i^*\}_{i \in I}$ is a set consisting of disjoint non-empty open sets of Y , which shows that $w(Y) = \aleph_1$.

Finally, let G be a non-empty G_δ -set in Y . First suppose that $\Omega \notin G$, and let $M = \bigcup_{i \in J} X_i$ be such that $G \cap M^* \neq \emptyset$, for some countable $J \subseteq I$. Now $G \cap M^*$ is a non-empty G_δ -set in $M^* \simeq \beta M \setminus M$. Theorem 1.2.5 of [15] states that each non-empty G_δ -set in the Stone-Čech remainder of a locally compact σ -compact space has infinite interior. Therefore since M is locally compact σ -compact, $G \cap M^*$ has non-empty interior in M^* . But M^* itself is open in Y , which implies that $\text{int}_Y G$ is infinite. Suppose that $\Omega \in G$. Then we can write $G = Y \setminus \bigcup_{n < \omega} C_n$, where each C_n

is a compact subset of $\sigma X \setminus X$. Let $P = \bigcup_{i \in J} X_i$ be such that $\bigcup_{n < \omega} C_n \subseteq \text{cl}_{\beta X} P$, where $J \subseteq I$ is countable. Choose a countable $K \subseteq I \setminus J$, and let $Q = \bigcup_{i \in K} X_i$. Then $Q^* \subseteq Y \setminus \text{cl}_{\beta X} P \subseteq G$. But Q contains a copy S of ω as a closed subset, and since Q^* is open in Y , we have $\omega^* \simeq S^* \subseteq Q^* \subseteq \text{int}_Y G$. Now Parovičenko Theorem completes the proof. \square

As it is noted in Theorem 5.7 of [9], for a zero-dimensional locally compact non-compact separable metrizable space X , the answer to the question of whether or not $\mathcal{E}_K(X)$ and $\mathcal{E}_K(\omega)$ are order-isomorphic depends on which model of set theory is being assumed. In the following we show that assuming the Continuum Hypothesis, if $w(X) = \aleph_1$, then $\mathcal{E}_K(X)$ and $\mathcal{E}_K(D(\aleph_1))$ are order-isomorphic (here $D(\aleph_1)$ denotes the discrete space of cardinality \aleph_1).

Theorem 4.4. [CH] *Let X and Y be zero-dimensional locally compact metrizable spaces of weights \aleph_1 . Then $\mathcal{E}_K(X)$ and $\mathcal{E}_K(Y)$ are order-isomorphic.*

Proof. By the above theorem, $S = (\sigma X \setminus X) \cup \{\Omega\} \simeq \omega^*$. We claim that Ω is in fact a P -point of S . So suppose that $\Omega \in Z \in \mathcal{Z}(S)$. Then $S \setminus Z \subseteq \sigma X \setminus X$ being a cozero-set in S is σ -compact, and therefore, assuming the notations of Theorem 1.1, there exists an $M = \bigcup_{i \in J} X_i$, where $J \subseteq I$ is countable, such that $S \setminus Z \subseteq M^*$. Therefore $S \setminus \text{cl}_{\beta X} M$ is an open neighborhood of Ω contained in Z . Similarly, Ω' is a P -point of $T = (\sigma Y \setminus Y) \cup \{\Omega'\} \simeq \omega^*$. By W. Rudin's Theorem, under [CH], for any two P -points of ω^* , there is a homeomorphism of ω^* onto itself which maps one point to another (see Theorem 7.11 of [16]). Let $f : S \rightarrow T$ be a homeomorphism such that $f(\Omega) = \Omega'$. Now arguing as in the proof of Theorem 4.2, we can show that $\mathcal{E}_K(X)$ and $\mathcal{E}_K(Y)$ are order-isomorphic. \square

Clearly since $\omega \subseteq D(\aleph_1)$, $\mathcal{E}_K(D(\aleph_1))$ contains an order-isomorphic copy of $\mathcal{E}_K(\omega)$. What is more interesting is the converse to this which is the subject of the next result.

Corollary 4.5. [CH] *There is an order isomorphism from $\mathcal{E}_K(D(\aleph_1))$ onto a subset of $\mathcal{E}_K(\omega)$. Such an order isomorphism is never onto $\mathcal{E}_K(\omega)$.*

Proof. Let $X = D(\aleph_1)$. By Theorem 4.3, we have $(\sigma X \setminus X) \cup \{\Omega\} \simeq \omega^*$. Let $f : (\sigma X \setminus X) \cup \{\Omega\} \rightarrow \omega^*$ be a homeomorphism. Let $Z \in \lambda_X(\mathcal{E}_K(X))$. Then by Lemma 6.6 of [9], we have $Z \subseteq \sigma X \setminus X$, and therefore $Z \subseteq M^*$, for some countable $M \subseteq X$. Now Z being clopen in X^* is clopen in M^* , and therefore it is clopen in $(\sigma X \setminus X) \cup \{\Omega\}$. Thus $f(Z)$ is a clopen subset of ω^* , i.e., $f(Z) \in \lambda_\omega(\mathcal{E}_K(\omega))$. Let $F : \lambda_X(\mathcal{E}_K(X)) \rightarrow \lambda_\omega(\mathcal{E}_K(\omega))$ be defined by $F(Z) = f(Z)$. Then F is clearly an order-isomorphism of $\lambda_X(\mathcal{E}_K(X))$ onto its image.

To show the second part of the theorem, we note that by Theorem 3.12, $\mathcal{E}_K(X)$ has no minimum whereas $\mathcal{E}_K(\omega)$ does. \square

We summarize the above results in the following theorem.

Theorem 4.6. *Let X and Y be zero-dimensional locally compact non-separable metrizable spaces. Then condition (1) implies the others.*

- (1) $\mathcal{E}_K(X)$ and $\mathcal{E}_K(Y)$ are order-isomorphic;
- (2) $\mathcal{Z}(\omega \sigma X \setminus X)$ and $\mathcal{Z}(\omega \sigma Y \setminus Y)$ are order-isomorphic;
- (3) The Boolean algebras of clopen sets $\mathcal{B}(\omega \sigma X \setminus X)$ and $\mathcal{B}(\omega \sigma Y \setminus Y)$ are order-isomorphic;

(4) $\omega\sigma X \setminus X$ and $\omega\sigma Y \setminus Y$ are homeomorphic.

Furthermore, if at least one of X and Y has weight greater than 2^{\aleph_0} , then the above conditions are equivalent. If we assume [CH], then the above conditions are all equivalent.

5. ON A SUBSET $\mathcal{E}_S(X)$ OF $\mathcal{E}(X)$

In this section we introduce a subset $\mathcal{E}_S(X)$ of $\mathcal{E}(X)$ and investigate its properties and its relation to the sets $\mathcal{E}_K(X)$ and $\mathcal{E}(X)$ introduced before.

Definition 5.1. For a locally compact non-separable metrizable space X , let

$$\mathcal{E}_S(X) = \{Y = X \cup \{p\} \in \mathcal{E}(X) : p \text{ has a separable neighborhood in } Y\}.$$

Theorem 5.2. For a locally compact non-separable metrizable space X we have

$$\mathcal{E}_S(X) = \{Y = X \cup \{p\} \in \mathcal{E}(X) : p \text{ has a } \sigma\text{-compact neighborhood in } Y\}.$$

Proof. Suppose that p has a σ -compact neighborhood W in $Y = X \cup \{p\} \in \mathcal{E}(X)$. Then W being a union of countably many compact (and therefore separable) subsets is separable, and so $Y \in \mathcal{E}_S(X)$.

To show the converse, let $\{U_n\}_{n < \omega}$ be the extension trace in X corresponding to $Y = X \cup \{p\} \in \mathcal{E}_S(X)$. Then there exists a $k < \omega$ such that $V = U_k \cup \{p\}$ is separable. Now since U_k is locally compact and separable, it is σ -compact (see 3.8.C of [7]). \square

In the following we first characterize the elements of $\mathcal{E}_S(X)$ in terms of their corresponding extension traces.

Lemma 5.3. Let X be a locally compact non-separable metrizable space and let $Y = X \cup \{p\} \in \mathcal{E}(X)$. Then the following conditions are equivalent.

- (1) $Y \in \mathcal{E}_S(X)$;
- (2) For every extension trace $\mathcal{U} = \{U_n\}_{n < \omega}$ in X generating Y , there exists a $k < \omega$ such that for every $n \geq k$, $\text{cl}_X U_n \setminus U_{n+1}$ is σ -compact;
- (3) There exists an extension trace $\mathcal{U} = \{U_n\}_{n < \omega}$ in X generating Y , such that for every $n < \omega$, $\text{cl}_X U_n \setminus U_{n+1}$ is σ -compact.

Proof. (1) implies (2). Suppose that $Y = X \cup \{p\} \in \mathcal{E}_S(X)$, and let $\{U_n\}_{n < \omega}$ be an extension trace in X which generates Y . Since p has a σ -compact neighborhood in Y , there exist a $k < \omega$ such that $\text{cl}_X U_k \cup \{p\}$ is σ -compact, and therefore $\text{cl}_X U_n \setminus U_{n+1}$ is σ -compact, for every $n \geq k$. That (2) implies (3), and (3) implies (1) are trivial. \square

The next result shows how $\mathcal{E}_S(X)$ is related to $\mathcal{E}_K(X)$.

Theorem 5.4. Let X be a locally compact non-separable metrizable space. Then we have

$$\mathcal{E}_S(X) = \{Y \in \mathcal{E}(X) : Y \geq S \text{ for some } S \in \mathcal{E}_K(X)\}.$$

Proof. Suppose that $Y \in \mathcal{E}(X)$ and let $S \in \mathcal{E}_K(X)$ be such that $Y \geq S$. Let $\mathcal{U} = \{U_n\}_{n < \omega}$ and $\mathcal{V} = \{V_n\}_{n < \omega}$ be extension traces in X , corresponding to $Y = X \cup \{p\}$ and $S = X \cup \{q\}$, respectively. Since $S \in \mathcal{E}_K(X)$, there exists an $n < \omega$ such that

$\text{cl}_X V_n \cup \{q\}$ is compact. Since $Y \geq S$, \mathcal{U} is finer than \mathcal{V} , and therefore there exists a $k < \omega$ such that $U_k \subseteq V_n$. Now since

$$\text{cl}_X U_k = \bigcup_{i \geq n} (\text{cl}_X U_k \cap (\text{cl}_X V_i \setminus V_{i+1}))$$

$\text{cl}_X U_k \cup \{p\}$ is a σ -compact neighborhood of p in Y , i.e., $Y \in \mathcal{E}_S(X)$.

For the converse, let $Y \in \mathcal{E}_S(X)$ and let $\mathcal{U} = \{U_n\}_{n < \omega}$ be an extension traces in X generating Y , with $\text{cl}_X U_n \setminus U_{n+1}$ being σ -compact for all $n < \omega$. Since $\text{cl}_X U_1 = \bigcup_{n < \omega} (\text{cl}_X U_n \setminus U_{n+1})$, we have $\lambda(Y) \subseteq \text{cl}_{\beta X} U_1 \subseteq \sigma X$, and thus $Y \geq S$ for some $S \in \mathcal{E}_K(X)$. \square

The following, under [CH], describes $\mathcal{E}_S(X)$ order-theoretically as a subset of $\mathcal{E}(X)$.

Theorem 5.5. [CH] *Let X be a locally compact non-separable metrizable space and let $S \in \mathcal{E}(X)$. Then $S \in \mathcal{E}_S(X)$ if and only if*

$$|\{Y \in \mathcal{E}(X) : Y \geq S\}| \leq \aleph_1.$$

Proof. Suppose that $S \in \mathcal{E}(X)$ is such that $|\{Y \in \mathcal{E}(X) : Y \geq S\}| \leq \aleph_1$. Then arguing as in the proof of Theorem 3.6 we have $\lambda(S) \subseteq \sigma X$, and therefore $S \geq T$, for some $T \in \mathcal{E}_K(X)$.

Conversely, if $S \in \mathcal{E}_S(X)$, then there exists a $T \in \mathcal{E}_K(X)$ such that $S \geq T$. Now the result follows, as by the proof of Theorem 3.6 we have $|\{Y \in \mathcal{E}(X) : Y \geq T\}| \leq \aleph_1$. \square

Next we find the image of $\mathcal{E}_S(X)$ under λ . This will be used in the subsequent results.

Theorem 5.6. *Let X be a locally compact non-separable metrizable space. Then we have*

$$\lambda(\mathcal{E}_S(X)) = \{Z \in \mathcal{Z}(\omega\sigma X \setminus X) : \Omega \notin Z\} \setminus \{\emptyset\}.$$

Proof. Let $Y \in \mathcal{E}_S(X)$. Then there exists an $S \in \mathcal{E}_K(X)$ such that $Y \geq S$, and thus $\lambda(Y) \subseteq \lambda(S) \subseteq \sigma X$. Now since $\lambda(S)$ is clopen in $\omega\sigma X \setminus X$, we have $\lambda(Y) \in \mathcal{Z}(\omega\sigma X \setminus X)$.

Conversely, suppose that $\emptyset \neq Z \in \mathcal{Z}(\omega\sigma X \setminus X)$ and $\Omega \notin Z$. Then since $Z \subseteq \sigma X \setminus X$, we have $Z \subseteq \lambda(S)$, for some $S \in \mathcal{E}_K(X)$. Thus $Z = \lambda(Y)$ for some $Y \geq S$, i.e., $Y \in \mathcal{E}_S(X)$. \square

Combined with Theorem 3.7, the following theorem shows that whenever $w(X) \geq 2^{\aleph_0}$, the sets $\mathcal{E}(X)$, $\mathcal{E}_K(X)$ and $\mathcal{E}_S(X)$ have three distinct order structures.

Theorem 5.7. *Let X be a locally compact non-separable metrizable space. Then $\mathcal{E}_K(X)$ and $\mathcal{E}_S(X)$ are never order-isomorphic. If moreover $w(X) \geq 2^{\aleph_0}$, then $\mathcal{E}(X)$ and $\mathcal{E}_S(X)$ are never order-isomorphic.*

Proof. The first part of the theorem follows from an argument similar to that of Theorem 3.7.

To show the second part, suppose that $\phi : \mathcal{E}(X) \rightarrow \mathcal{E}_S(X)$ is an order-isomorphism. By part (4) of Theorem 3.19, there exists a sequence $\{Y_n\}_{n < \omega}$ in $\mathcal{E}(X)$ with no lower bound in $\mathcal{E}(X)$. Let for each $n < \omega$, $S_n \in \mathcal{E}_K(X)$ be such that $S_n \leq \phi(Y_n)$ (see Theorem 5.4). Then by part (5) of Theorem 3.19, the sequence $\{S_n\}_{n < \omega}$ and therefore the sequence $\{\phi(Y_n)\}_{n < \omega}$ has a lower bound in $\mathcal{E}_K(X)$, contradicting to our assumptions. \square

For a point x in a space X , we denote by $w(x, X)$, the smallest weight of an open neighborhood of x in X . In the following lemma, under [CH], we characterize the points of $\sigma X \setminus X$ in X^* .

Lemma 5.8. [CH] *Let X be a locally compact non-separable metrizable space. Then the set $\sigma X \setminus X$ consists of exactly those elements $x \in X^*$ for which $w(x, X^*) \leq \aleph_1$.*

Proof. Assume the notations of Theorem 1.1. Let $x \in \sigma X \setminus X$. Then $x \in M^*$, where $M = \bigcup_{i \in J} X_i$, for some countable $J \subseteq I$. Since M is separable, we have $w(M^*) \leq \aleph_1$.

Conversely, suppose that $x \in X^*$ is such that $w(x, X^*) \leq \aleph_1$. Suppose that $x \notin \sigma X$. Let V be an open neighborhood of x in X^* such that $w(\text{cl}_{X^*} V) \leq \aleph_1$. Let $f \in C(X^*, \mathbf{I})$ be such that $f(x) = 0$ and $f(X^* \setminus V) \subseteq \{1\}$. We show that there exists a $Z \in \lambda(\mathcal{E}(X))$ such that $Z \subseteq S = Z(f)$ and $Z \setminus \sigma X \neq \emptyset$. Since $S \in \mathcal{Z}(X^*)$, by Lemma 4.2 of [9], there exists a regular sequence of open sets $\mathcal{U} = \{U_n\}_{n < \omega}$ in X such that $S = \bigcap_{n < \omega} U_n^*$. Let

$$L = \left\{ i \in I : X_i \cap \bigcap_{n < \omega} U_n \neq \emptyset \right\}.$$

We consider the following two cases.

Case 1) Suppose that L is countable. Let for each $n < \omega$, $V_n = U_n \setminus \bigcup_{i \in L} X_i$. Then since

$$\bigcap_{n < \omega} V_n = \bigcap_{n < \omega} U_n \setminus \bigcup_{i \in L} X_i = \emptyset$$

$\mathcal{V} = \{V_n\}_{n < \omega}$ is an extension trace in X . Now for each $n < \omega$, we have

$$U_n^* = \left(U_n \cap \bigcup_{i \in L} X_i \right)^* \cup \left(U_n \setminus \bigcup_{i \in L} X_i \right)^* \subseteq (\sigma X \setminus X) \cup V_n^*$$

and therefore since $x \in S = \bigcap_{n < \omega} U_n^* \setminus \sigma X$, we have $x \in \bigcap_{n < \omega} V_n^*$. Let $Z = \bigcap_{n < \omega} V_n^* \in \lambda(\mathcal{E}(X))$. Then clearly $Z \subseteq S$ and $Z \setminus \sigma X \neq \emptyset$.

Case 2) Suppose that L is uncountable. Let $\{L_n\}_{n < \omega}$ be a partition of L into mutually disjoint uncountable subsets. Let for each $n < \omega$

$$V_n = U_n \cap \bigcup \{X_i : i \in L_n \cup L_{n+1} \cup \dots\}.$$

Then $\mathcal{V} = \{V_n\}_{n < \omega}$ is an extension trace in X . We verify that for each $n < \omega$, $\text{cl}_{\beta X} V_n \setminus \sigma X \neq \emptyset$. For otherwise, if for some $n < \omega$, $\text{cl}_{\beta X} V_n \subseteq \sigma X$, then $\text{cl}_{\beta X} V_n \subseteq \text{cl}_{\beta X} (\bigcup_{i \in H} X_i)$, for some countable $H \subseteq I$, and therefore $V_n \subseteq \bigcup_{i \in H} X_i$, which is a contradiction, as each L_n is chosen to be uncountable. By compactness of βX , we have $\bigcap_{n < \omega} (\text{cl}_{\beta X} V_n \setminus \sigma X) \neq \emptyset$. Let in this case $Z = \bigcap_{n < \omega} V_n^* \in \lambda(\mathcal{E}(X))$. Then clearly $Z \subseteq S$ and $Z \setminus \sigma X \neq \emptyset$.

Let $A \in \mathcal{E}(X)$ be such that $Z = \lambda(A)$. By Theorem 5.6, $A \notin \mathcal{E}_S(X)$ and thus by Theorem 5.5, $|\{Y \in \mathcal{E}(X) : Y \geq A\}| > \aleph_1$. Lemma 15.19 of [6] states that for a σ -compact space T with $w(T) \leq 2^{\aleph_0}$, we have $|C(T)| \leq 2^{\aleph_0}$. Now applying this to $\text{cl}_{X^*} V$, we obtain $|\mathcal{Z}(\text{cl}_{X^*} V)| \leq \aleph_1$, which is a contradiction. This proves our lemma. \square

Theorem 5.9. [CH] *Let X and Y be locally compact non-separable metrizable spaces. If X^* and Y^* are homeomorphic, then $\mathcal{E}_S(X)$ and $\mathcal{E}_S(Y)$ are order-isomorphic.*

Proof. By Lemma 5.8 any homeomorphism between X^* and Y^* induces a homeomorphism between $\sigma X \setminus X$ and $\sigma Y \setminus Y$. Now the proof is completed by a slight modification of the last part of the proof of Theorem 4.2. \square

The following is analogous to Theorems 4.1 and 4.2. It shows how the order structure of $\mathcal{E}_S(X)$ and the topology of $\sigma X \setminus X$ are related to each other.

Theorem 5.10. *Let X and Y be locally compact non-separable metrizable spaces and let $\omega\sigma X = \sigma X \cup \{\Omega\}$ and $\omega\sigma Y = \sigma Y \cup \{\Omega'\}$ be the one-point compactifications of σX and σY , respectively. If $\mathcal{E}_S(X)$ and $\mathcal{E}_S(Y)$ are order-isomorphic, then $\omega\sigma X \setminus X$ and $\omega\sigma Y \setminus Y$ are homeomorphic.*

Proof. Let $\phi : \lambda_X(\mathcal{E}_S(X)) \rightarrow \lambda_Y(\mathcal{E}_S(Y))$ be an order-isomorphism. We extend ϕ by letting $\phi(\emptyset) = \emptyset$. We define a function $\psi : \mathcal{Z}(\omega\sigma X \setminus X) \rightarrow \mathcal{Z}(\omega\sigma Y \setminus Y)$ and verify that it is an order-isomorphism.

For a $Z \in \mathcal{Z}(\omega\sigma X \setminus X)$, with $\Omega \notin Z$, let $\psi(Z) = \phi(Z)$.

Now suppose that $Z \in \mathcal{Z}(\omega\sigma X \setminus X)$ and $\Omega \in Z$. Then $(\omega\sigma X \setminus X) \setminus Z$, being a cozero-set in $\omega\sigma X \setminus X$, can be written as $(\omega\sigma X \setminus X) \setminus Z = \bigcup_{n < \omega} Z_n$, where for each $n < \omega$, $Z_n \in \mathcal{Z}(\omega\sigma X \setminus X)$ and $\Omega \notin Z_n$, and thus by Theorem 5.6, $Z_n \in \lambda_X(\mathcal{E}_S(X))$. We claim that $\bigcup_{n < \omega} \phi(Z_n)$ is a cozero-set in $\omega\sigma Y \setminus Y$. To show this, let $Y = \bigoplus_{i \in J} Y_i$, with each Y_i being a separable non-compact subspace. Since for each $n < \omega$, $\phi(Z_n) \subseteq \sigma Y \setminus Y$, there exists a countable $L \subseteq J$ such that $\bigcup_{n < \omega} \phi(Z_n) \subseteq (\bigcup_{i \in L} Y_i)^* = \phi(A)$, for some $A \in \lambda_X(\mathcal{E}_S(X))$. We show that $\phi(A \cap Z) = \phi(A) \setminus \bigcup_{n < \omega} \phi(Z_n)$. Since for each $n < \omega$, $A \cap Z \cap Z_n = \emptyset$, we have $\phi(A \cap Z) \cap \phi(Z_n) = \emptyset$, and therefore $\phi(A \cap Z) \subseteq \phi(A) \setminus \bigcup_{n < \omega} \phi(Z_n)$. To show the converse, let $x \in \phi(A) \setminus \bigcup_{n < \omega} \phi(Z_n)$. Since for each $n < \omega$, $x \notin \phi(Z_n)$, there exists a $B \in \mathcal{Z}(\omega\sigma Y \setminus Y)$ such that $x \in B$, and for each $n < \omega$, $B \cap \phi(Z_n) = \emptyset$. If $x \notin \phi(A \cap Z)$, then there exists a $C \in \mathcal{Z}(\omega\sigma Y \setminus Y)$ such that $x \in C$ and $C \cap \phi(A \cap Z) = \emptyset$. Consider $D = \phi(A) \cap B \cap C \in \lambda_Y(\mathcal{E}_S(Y))$, and let $E \in \lambda_X(\mathcal{E}_S(X))$ be such that $\phi(E) = D$. Then since for each $n < \omega$, $\phi(E) \cap \phi(Z_n) = \emptyset$, we have $E \cap Z_n = \emptyset$, and therefore $E \subseteq Z$. On the other hand since $\phi(E) \subseteq \phi(A)$, we have $E \subseteq A$ and thus $E \subseteq A \cap Z$. Therefore, $\phi(E) \subseteq \phi(A \cap Z)$, which implies that $\phi(E) = \emptyset$, as $\phi(E) \subseteq C$. This contradiction shows that $x \in \phi(A \cap Z)$, and therefore $\phi(A \cap Z) = \phi(A) \setminus \bigcup_{n < \omega} \phi(Z_n)$. Now since $\phi(A)$ is clopen in $\sigma Y \setminus Y$, we have

$$\begin{aligned} (\omega\sigma Y \setminus Y) \setminus \bigcup_{n < \omega} \phi(Z_n) &= (\phi(A) \setminus \bigcup_{n < \omega} \phi(Z_n)) \cup ((\omega\sigma Y \setminus Y) \setminus \phi(A)) \\ &= \phi(A \cap Z) \cup ((\omega\sigma Y \setminus Y) \setminus \phi(A)) \in \mathcal{Z}(\omega\sigma Y \setminus Y) \end{aligned}$$

and our claim is verified. In this case we define $\psi(Z) = (\omega\sigma Y \setminus Y) \setminus \bigcup_{n < \omega} \phi(Z_n)$.

Next we show that ψ is well-defined. So assume another representation for Z , i.e., suppose that $Z = (\omega\sigma X \setminus X) \setminus \bigcup_{n < \omega} S_n$, with $S_n \in \lambda_X(\mathcal{E}_S(X)) \cup \{\emptyset\}$, for all $n < \omega$. Suppose that $\bigcup_{n < \omega} \phi(Z_n) \neq \bigcup_{n < \omega} \phi(S_n)$. Without any loss of generality we may assume that $\bigcup_{n < \omega} \phi(Z_n) \setminus \bigcup_{n < \omega} \phi(S_n) \neq \emptyset$. Let $x \in \bigcup_{n < \omega} \phi(Z_n) \setminus \bigcup_{n < \omega} \phi(S_n)$. Let $m < \omega$ be such that $x \in \phi(Z_m)$. Then since $x \notin \bigcup_{n < \omega} \phi(S_n)$, there exists an $A \in \mathcal{Z}(\omega\sigma Y \setminus Y)$ such that $x \in A$ and $A \cap \bigcup_{n < \omega} \phi(S_n) = \emptyset$. Consider $A \cap \phi(Z_m) \in \lambda_Y(\mathcal{E}_S(Y))$. Let $B \in \lambda_X(\mathcal{E}_S(X))$ be such that $\phi(B) = A \cap \phi(Z_m)$. Since $\phi(B) \subseteq A$, we have $B \cap S_n = \emptyset$, for all $n < \omega$. But $B \subseteq Z_m \subseteq \bigcup_{n < \omega} Z_n = \bigcup_{n < \omega} S_n$, which implies that $B = \emptyset$, which is a contradiction. Therefore $\bigcup_{n < \omega} \phi(Z_n) = \bigcup_{n < \omega} \phi(S_n)$, and thus ψ is well defined.

To prove that ψ is an order-isomorphism, let $S, Z \in \mathcal{Z}(\omega\sigma X \setminus X)$ with $S \subseteq Z$. Assume that $S \neq \emptyset$. We consider the following three cases.

Case 1) Suppose that $\Omega \notin Z$. Then $\psi(S) = \phi(S) \subseteq \phi(Z) = \psi(Z)$.

Case 2) Suppose that $\Omega \notin S$ and $\Omega \in Z$. Let $Z = (\omega\sigma X \setminus X) \setminus \bigcup_{n < \omega} Z_n$, with $Z_n \in \lambda_X(\mathcal{E}_S(X)) \cup \{\emptyset\}$, for all $n < \omega$. Then since $S \subseteq Z$, for each $n < \omega$, we have $S \cap Z_n = \emptyset$, and therefore $\phi(S) \cap \phi(Z_n) = \emptyset$. We have

$$\psi(S) = \phi(S) \subseteq (\omega\sigma Y \setminus Y) \setminus \bigcup_{n < \omega} \phi(Z_n) = \psi(Z).$$

Case 3) Suppose that $\Omega \in S$ and let

$$Z = (\omega\sigma X \setminus X) \setminus \bigcup_{n < \omega} Z_n \text{ and } S = (\omega\sigma X \setminus X) \setminus \bigcup_{n < \omega} S_n$$

where for each $n < \omega$, $S_n, Z_n \in \lambda_X(\mathcal{E}_S(X)) \cup \{\emptyset\}$. Since $S \subseteq Z$ we have $\bigcup_{n < \omega} Z_n \subseteq \bigcup_{n < \omega} S_n$, and so $S = (\omega\sigma X \setminus X) \setminus \bigcup_{n < \omega} (S_n \cup Z_n)$. Therefore

$$\psi(S) = (\omega\sigma Y \setminus Y) \setminus \bigcup_{n < \omega} (\phi(S_n) \cup \phi(Z_n)) \subseteq (\omega\sigma Y \setminus Y) \setminus \bigcup_{n < \omega} \phi(Z_n) = \psi(Z)$$

and thus ψ is an order-homomorphism.

To show that ψ is an order-isomorphism, we note that $\phi^{-1} : \lambda_Y(\mathcal{E}_S(Y)) \rightarrow \lambda_X(\mathcal{E}_S(X))$ is an order-isomorphism. Let $\gamma : \mathcal{Z}(\omega\sigma Y \setminus Y) \rightarrow \mathcal{Z}(\omega\sigma X \setminus X)$ be its induced order-homomorphism defined as above. Then it is straightforward to see that $\gamma = \psi^{-1}$, i.e., ψ is an order-isomorphism and thus $\mathcal{Z}(\omega\sigma X \setminus X)$ and $\mathcal{Z}(\omega\sigma Y \setminus Y)$ are order-isomorphic, which implies that $\omega\sigma X \setminus X$ and $\omega\sigma Y \setminus Y$ are homeomorphic. \square

The next result is the converse of the above theorem under some weight restrictions.

Theorem 5.11. *Let X and Y be locally compact non-separable metrizable spaces. Suppose moreover that at least one of X and Y has weight greater than 2^{\aleph_0} . Let $\omega\sigma X$ and $\omega\sigma Y$ be as in the above theorem. Then if $\omega\sigma X \setminus X$ and $\omega\sigma Y \setminus Y$ are homeomorphic, $\mathcal{E}_S(X)$ and $\mathcal{E}_S(Y)$ are order-isomorphic.*

Proof. This follows by a slight modification of the proof of Theorem 4.2. \square

We summarize the above theorems as follows.

Theorem 5.12. *Let X and Y be locally compact non-separable metrizable spaces. Suppose moreover that at least one of X and Y has weight greater than 2^{\aleph_0} . Then the following conditions are equivalent.*

- (1) $\mathcal{E}_S(X)$ and $\mathcal{E}_S(Y)$ are order-isomorphic;
- (2) $\mathcal{Z}(\omega\sigma X \setminus X)$ and $\mathcal{Z}(\omega\sigma Y \setminus Y)$ are order-isomorphic;
- (3) $\omega\sigma X \setminus X$ and $\omega\sigma Y \setminus Y$ are homeomorphic.

Comparing Theorems 4.6 and 5.12, we deduce that for zero-dimensional locally compact non-separable metrizable spaces X and Y , such that at least one of them has weight greater than 2^{\aleph_0} , $\mathcal{E}_S(X)$ and $\mathcal{E}_S(Y)$ are order-isomorphic if and only if $\mathcal{E}_K(X)$ and $\mathcal{E}_K(Y)$ are. It turns out that even more is true.

Theorem 5.13. *Let X and Y be zero-dimensional locally compact non-separable metrizable spaces and let $f : \mathcal{E}_K(X) \rightarrow \mathcal{E}_K(Y)$ be an order-isomorphism. Then there exists an order-isomorphism $F : \mathcal{E}_S(X) \rightarrow \mathcal{E}_S(Y)$ such that $F|_{\mathcal{E}_K(X)} = f$.*

Proof. Let $g = \lambda_Y f \lambda_X^{-1} : \lambda_X(\mathcal{E}_K(X)) \rightarrow \lambda_Y(\mathcal{E}_K(Y))$ and $G : \mathcal{B}(\omega\sigma X \setminus X) \rightarrow \mathcal{B}(\omega\sigma Y \setminus Y)$ be as defined in the proof of Theorem 4.1. Then as it is shown there, G is an order-isomorphism, and since $\omega\sigma X \setminus X$ and $\omega\sigma Y \setminus Y$ are zero-dimensional, there exists a homeomorphism $\phi : \omega\sigma X \setminus X \rightarrow \omega\sigma Y \setminus Y$ such that $\phi(U) = G(U)$, for any $U \in \mathcal{B}(\omega\sigma X \setminus X)$. Let $H : \lambda_X(\mathcal{E}_S(X)) \rightarrow \lambda_Y(\mathcal{E}_S(Y))$ be defined by $H(Z) = \phi(Z)$. We verify that H is a well-defined order-isomorphism.

First we note that $\phi(\Omega) = \Omega'$. For otherwise, if $\phi(x) = \Omega'$, for some $x \neq \Omega$, then since $x \in \sigma X \setminus X$, assuming the notations of Theorem 1.1, we have $x \in (\bigcup_{i \in L} X_i)^* = U$, for some countable $L \subseteq I$. Now since U is clopen in $\omega\sigma X \setminus X$, we have $\phi(U) = G(U)$, and by the way we defined G , $G(U) = g(U) \in \lambda_Y(\mathcal{E}_K(Y))$. But this implies that $\Omega' = \phi(x) \in \phi(U) \in \lambda_Y(\mathcal{E}_K(Y))$, which is a contradiction. Therefore $\phi(\Omega) = \Omega'$.

Now suppose that $Z \in \lambda_X(\mathcal{E}_S(X))$. Then by Theorem 5.6, $Z \in \mathcal{Z}(\omega\sigma X \setminus X)$ and $\Omega \notin Z$. Therefore $\phi(Z)$ is a zero-set in $\omega\sigma Y \setminus Y$ such that $\Omega' \notin \phi(Z)$. This shows that H is well-defined. By the way we defined H , it is clearly an order-isomorphism. Now let $U \in \lambda_X(\mathcal{E}_K(X))$. Then since $U \in \mathcal{B}(\omega\sigma X \setminus X)$, we have $H(U) = \phi(U) = G(U)$. But by definition of G , since $\Omega \notin U$, $G(U) = g(U)$, and therefore $H(U) = g(U)$, i.e., $H|_{\lambda_X(\mathcal{E}_K(X))} = g$. Now let $F = \lambda_Y^{-1} H \lambda_X : \mathcal{E}_S(X) \rightarrow \mathcal{E}_S(Y)$. Clearly F is an order-isomorphism and by definition of g , for any $A \in \mathcal{E}_K(X)$, we have $F(A) = \lambda_Y^{-1} H \lambda_X(A) = \lambda_Y^{-1} g \lambda_X(A) = f(A)$, i.e., $F|_{\mathcal{E}_K(X)} = f$ and the proof is complete. \square

The next result gives an order-theoretic characterization of $\mathcal{E}_K(X)$ as a subset of $\mathcal{E}_S(X)$.

Theorem 5.14. *Let X be a locally compact non-separable metrizable spaces. For a set $\mathcal{F} \subseteq \mathcal{E}_S(X)$ consider the following conditions.*

- (1) *For each $A \in \mathcal{E}_S(X)$, there exists a $B \in \mathcal{F}$ such that $B \leq A$;*
- (2) *For each $A, B \in \mathcal{F}$ such that $A < B$, there exists a $C \in \mathcal{F}$ such that $B \wedge C = A$, and B and C have no common upper bound in $\mathcal{E}_S(X)$.*

Then the set $\mathcal{E}_K(X)$ is the largest (with respect to set-theoretic inclusion) subset of $\mathcal{E}_S(X)$ satisfying the above conditions.

Proof. We first check that $\mathcal{E}_K(X)$ satisfies the above conditions. Condition (1) follows from Theorem 5.4. To show that $\mathcal{E}_K(X)$ satisfies condition (2), suppose that $A, B \in \mathcal{E}_K(X)$, with $A < B$. Let $\lambda(C) = \lambda(A) \setminus \lambda(B)$, for some $C \in \mathcal{E}_K(X)$. Clearly $C \geq A$, and if $Y \in \mathcal{E}_S(X)$ is such that $C \geq Y$ and $B \geq Y$, then $A \geq Y$. Thus $A = B \wedge C$. It is clear that B and C have no common upper bound in $\mathcal{E}_S(X)$.

Now suppose that $\mathcal{F} \subseteq \mathcal{E}_S(X)$ satisfies conditions (1) and (2). Let $Y \in \mathcal{F}$. Then by Theorem 5.6, we have $\lambda(Y) \subseteq \sigma X \setminus X$. Assume the notations of Theorem 1.1. Then $\lambda(Y) \subseteq (\bigcup_{i \in J} X_i)^* = \lambda(A)$ (properly), for some countable $J \subseteq I$. Using condition (1), let $B \in \mathcal{F}$ be such that $B \leq A$. Then since $Y > B$, by condition (2), there exists a $C \in \mathcal{F}$ such that $Y \wedge C = B$ and Y, C have no common upper bound in $\mathcal{E}_S(X)$. Let $D \in \mathcal{E}_S(X)$ be such that $\lambda(D) = \lambda(Y) \cup \lambda(C)$. Then since $D \leq Y$ and $D \leq C$ we have $D \leq B$. Also since $Y \geq B$ and $C \geq B$, we have $\lambda(B) \supseteq \lambda(D)$, and therefore $B = D$. Now as Y and C have no upper bound in common $\lambda(Y) \cap \lambda(C) = \emptyset$, and therefore as $\lambda(Y) = \lambda(A) \setminus (\lambda(A) \cap \lambda(C))$, $\lambda(Y)$ is a clopen subset of X^* . Thus $Y \in \mathcal{E}_K(X)$ and therefore $\mathcal{F} \subseteq \mathcal{E}_K(X)$. \square

Theorem 5.15. *Let X and Y be locally compact non-separable metrizable spaces. Then for any order-isomorphism $\phi : \mathcal{E}_S(X) \rightarrow \mathcal{E}_S(Y)$ we have $\phi(\mathcal{E}_K(X)) = \mathcal{E}_K(Y)$.*

Proof. Let $\mathcal{F} = \phi(\mathcal{E}_K(X))$. Then it is easy to see that \mathcal{F} satisfies the conditions of Theorem 5.14, and thus by maximality $\mathcal{F} \subseteq \mathcal{E}_K(Y)$. The reverse inclusion holds by symmetry. \square

The following result is analogous to Theorem 4.4, replacing $\mathcal{E}_K(X)$ and $\mathcal{E}_K(Y)$ by $\mathcal{E}_S(X)$ and $\mathcal{E}_S(Y)$, respectively.

Theorem 5.16. [CH] *Let X and Y be zero-dimensional locally compact metrizable spaces of weights \aleph_1 . Then $\mathcal{E}_S(X)$ and $\mathcal{E}_S(Y)$ are order-isomorphic.*

Proof. By Theorem 4.4, $\mathcal{E}_K(X)$ and $\mathcal{E}_K(Y)$ are order-isomorphic. The result now follows as by Theorem 5.13 every such order-isomorphism can be extended to an order-isomorphism of $\mathcal{E}_S(X)$ onto $\mathcal{E}_S(Y)$. \square

The following example shows that zero-dimensionality cannot be omitted from Theorems 5.16 and 4.4.

Example 5.17. Let $X = D(\aleph_1)$ and $Y = \bigoplus_{i < \omega_1} Y_i$, where for each $i < \omega_1$, $Y_i = \mathbf{R}$. Suppose that $\mathcal{E}_S(X)$ and $\mathcal{E}_S(Y)$ are order-isomorphic, and let $\phi : \mathcal{E}_S(X) \rightarrow \mathcal{E}_S(Y)$ denote an order-isomorphism. By Theorem 6.1, $\phi|_{\mathcal{E}_K(X)} : \mathcal{E}_K(X) \rightarrow \mathcal{E}_K(Y)$ is also an order-isomorphism. Let $j < \omega_1$ and let $T \in \mathcal{E}_K(X)$ be such that $\lambda(T) = Y_j^*$. Let $S \in \mathcal{E}_K(X)$ be such that $\phi(S) = T$. Then since the number of clopen subsets of $Y_j^* = \mathbf{R}^*$ is finite, there are only finitely many $A \in \mathcal{E}_K(X)$ such that $A \geq S$, which is a contradiction, as $\lambda_X(S) \subseteq D^* \simeq \omega^*$, for some countable $D \subseteq X$, and there are infinitely many clopen subsets of $\lambda_X(S)$ each corresponding to an element $A \in \mathcal{E}_K(X)$ with $A \geq S$.

In Theorems 3.6 and 5.14, we characterized $\mathcal{E}_K(X)$ among the subsets of $\mathcal{E}(X)$ and $\mathcal{E}_S(X)$, respectively. In the following we give a characterization of $\mathcal{E}_S(X)$ among the subsets of $\mathcal{E}(X)$ which contain $\mathcal{E}_K(X)$.

Theorem 5.18. *Let X be a zero-dimensional locally compact non-separable metrizable space. Then the set $\mathcal{E}_S(X)$ is the smallest (with respect to set-theoretic inclusion) subset of $\mathcal{E}(X)$ containing $\mathcal{E}_K(X)$, such that for every upper bounded (in $\mathcal{E}(X)$) sequence in $\mathcal{E}_K(X)$, it contains its least upper bound (in $\mathcal{E}(X)$).*

Proof. Using Theorem 5.4, it can be seen that the set $\mathcal{E}_S(X)$ satisfies the above requirements.

Now suppose that $\mathcal{E}_K(X) \subseteq \mathcal{F} \subseteq \mathcal{E}_S(X)$ satisfies the conditions of the theorem. Let $Y \in \mathcal{E}_S(X)$. By Lemma 3.17, we have $\lambda(Y) = \bigcap_{n < \omega} U_n^*$, for some extension trace $\{U_n\}_{n < \omega}$ in X consisting of clopen subsets of X . Since $Y \in \mathcal{E}_S(X)$, by Theorem 5.4, $Y \geq S$ for some $S \in \mathcal{E}_K(X)$. Now for any $n < \omega$, $\lambda(S) \cap U_n^*$ is a clopen subset of X^* , and therefore $\lambda(S) \cap U_n^* = \lambda(Y_n)$, for some $Y_n \in \mathcal{E}_K(X)$. Clearly for any $n < \omega$, $\lambda(Y) \subseteq \lambda(Y_n)$, and thus $Y \geq Y_n$. Therefore $Y = \bigvee_{n < \omega} Y_n$, and thus by assumption $Y \in \mathcal{F}$, which shows that $\mathcal{E}_S(X) \subseteq \mathcal{F}$. \square

6. SOME CARDINALITY THEOREMS

In this section we obtain some theorems on the cardinality of the sets $\mathcal{E}_K(X)$ and $\mathcal{E}(X)$. By modifying the proofs, similar results can be obtained on the cardinality of the set $\mathcal{E}_S(X)$.

Theorem 6.1. *Let X be a locally compact non-compact metrizable space. Then we have*

$$|\mathcal{E}(X)| = 2^{w(X)}.$$

Proof. First suppose that X is separable. By Theorem 5.3 of [9], $\mathcal{E}(X)$ and $\mathcal{Z}(X^*) \setminus \{\emptyset\}$ are order-anti-isomorphic. Since X is non-compact, it contains a copy M of ω as a closed subset, and therefore since $\omega^* \simeq \text{cl}_{\beta X} M \setminus M$, we may assume that $\omega^* \subseteq X^*$. But ω^* is z -embedded in X^* and therefore

$$|\mathcal{E}(X)| = |\mathcal{Z}(X^*)| \geq |\mathcal{Z}(\omega^*)| = 2^{\aleph_0} = 2^{w(X)}.$$

Now suppose that X is non-separable and assume the notations of Theorem 1.1. Clearly $|I| = w(X)$. For each $i \in I$, let $\{U_n^i\}_{n < \omega}$ be an extension trace in X_i . For each non-empty $J \subseteq I$ and each $n < \omega$, let $V_J^n = \bigcup_{i \in J} U_n^i$. Then it is easy to see that $\mathcal{V}_J = \{V_J^n\}_{n < \omega}$ is an extension trace in X and \mathcal{V}_{J_1} and \mathcal{V}_{J_2} are non-equivalent for $J_1 \neq J_2$. Thus in this case $|\mathcal{E}(X)| \geq |\mathcal{P}(I)| = 2^{w(X)}$.

Finally, we note that to every extension trace $\{U_n\}_{n < \omega}$ in X , there corresponds a sequence $\{\mathcal{B}_n\}_{n < \omega}$ of subsets of \mathcal{B} , where \mathcal{B} is a base for X of cardinality $w(X)$, in such a way that $U_n = \bigcup \mathcal{B}_n$, for all $n < \omega$. Since the number of such sequences does not exceed $|\mathcal{P}(\mathcal{B})|^{\aleph_0} = 2^{w(X)}$, it follows that $2^{w(X)} \geq |\mathcal{E}(X)|$, and thus combined with above, this implies that equality holds. \square

By a known result of Tarski (Tarski Theorem) for any infinite set E , there is a collection \mathcal{A} of subsets of E such that $|\mathcal{A}| = |E|^{\aleph_0}$, $|A| = \aleph_0$ for any $A \in \mathcal{A}$ and the intersection of any two distinct elements of \mathcal{A} is finite (see Theorem 2.1 of [10]). We use this in the following theorem.

Theorem 6.2. *Let X be a locally compact non-compact metrizable space. Then we have*

$$|\mathcal{E}_K(X)| \leq w(X)^{\aleph_0}.$$

Furthermore, if X is non-separable or zero-dimensional, then equality holds.

Proof. Let $Y \in \mathcal{E}_K(X)$. Then by Lemma 3.1, there exists an extension trace $\mathcal{U} = \{U_n\}_{n < \omega}$ in X which generates Y , and $\text{cl}_X U_n \setminus U_{n+1}$ is compact for all $n < \omega$. Let \mathcal{B} be a base in X with $|\mathcal{B}| = w(X)$, and let $U_0 = X$. Then since for all $n < \omega$, $\text{cl}_X U_n \setminus U_{n+1}$ is a compact subset of the open set $U_{n-1} \setminus \text{cl}_X U_{n+2}$, it follows that there exists a $k_n < \omega$ and $C_1^n, \dots, C_{k_n}^n \in \mathcal{B}$ such that

$$\text{cl}_X U_n \setminus U_{n+1} \subseteq C_1^n \cup \dots \cup C_{k_n}^n \subseteq U_{n-1} \setminus \text{cl}_X U_{n+2}.$$

Let

$$V_n = \bigcup_{i=1}^{k_n} C_i^n \cup \bigcup_{i=1}^{k_{n+1}} C_i^{n+1} \cup \dots.$$

Then clearly $V_n \subseteq U_{n-1}$. On the other hand since $\bigcap_{n < \omega} U_n = \emptyset$, it follows from $\text{cl}_X U_j \setminus U_{j+1} \subseteq \bigcup_{i=1}^{k_j} C_i^j$, $j = n, n+1, \dots$ that $\text{cl}_X U_n \subseteq V_n$. Now $\mathcal{V} = \{V_n\}_{n < \omega}$ is an extension trace in X equivalent to \mathcal{U} , and therefore Y is also generated by \mathcal{V} . So to each $Y \in \mathcal{E}_K(X)$, there corresponds a sequence $\{\{C_i^n\}_{i=1}^{k_n}\}_{n < \omega}$ which consists of finite subsets of \mathcal{B} . Since the number of such sequences is not greater than $w(X)^{\aleph_0}$, we have $|\mathcal{E}_K(X)| \leq w(X)^{\aleph_0}$.

For the second part of the theorem we consider the following two cases.

Case 1) Suppose that X is separable and zero-dimensional. By Proposition 5.1 of [9], $X = \bigcup_{n < \omega} C_n$, where each C_n is open with compact closure in X , and

$\text{cl}_X C_n \subseteq C_{n+1}$ for all $n < \omega$. Let $U_n = X \setminus \text{cl}_X C_n$. Then clearly $\mathcal{U} = \{U_n\}_{n < \omega}$ is an extension trace in X . By Lemma 3.17 we can choose an extension trace $\mathcal{V} = \{V_n\}_{n < \omega}$ consisting of clopen subsets of X equivalent to \mathcal{U} . For each $n < \omega$, there exists a k_n such that $U_{k_n} \subseteq V_n$, and so $X \setminus V_n \subseteq X \setminus U_{k_n} = \text{cl}_X C_{k_n}$. Therefore $X \setminus V_n$ and thus $V_n \setminus V_{n+1}$ is compact. Let D_1, D_2, \dots be distinct non-empty sets of the form $V_n \setminus V_{n+1}$. Now let $\{N_t\}_{t < 2^{\aleph_0}}$ be a partition of ω into infinite almost disjoint subsets. For $t < 2^{\aleph_0}$ let $N_t = \{n_1^t, n_2^t, \dots\}$, where $n_i^t \neq n_j^t$, for distinct $i, j < \omega$, and let

$$\mathcal{V}_t = \{D_{n_k^t} \cup D_{n_{k+1}^t} \cup \dots\}_{k < \omega}$$

which is an extension trace of clopen subsets of X . Clearly each \mathcal{V}_t is corresponding to an elements of $\mathcal{E}_K(X)$, and since the corresponding members of $\mathcal{E}_K(X)$ are distinct we have $|\mathcal{E}_K(X)| \geq 2^{\aleph_0}$.

Case 2) Suppose that X is not separable and assume the notations of Theorem 1.1. By Tarski Theorem, there exists a collection \mathcal{J} of subsets of I , with $|\mathcal{J}| = |I|^{\aleph_0} = w(X)^{\aleph_0}$ and $|J| = \aleph_0$, for every $J \in \mathcal{J}$, such that the intersection of any two distinct elements of \mathcal{J} is finite. Let for each $J \in \mathcal{J}$, $Y_J \in \mathcal{E}_K(X)$ be such that $\lambda(Y_J) = (\bigcup_{i \in J} X_i)^*$. Then $\{Y_J : J \in \mathcal{J}\}$ is a collection of distinct elements of $\mathcal{E}_K(X)$ and therefore $|\mathcal{E}_K(X)| \geq w(X)^{\aleph_0}$. \square

7. SOME QUESTIONS

Assume [CH]. Let X and Y be zero-dimensional locally compact non-separable metrizable spaces. Suppose that $\phi : \mathcal{E}(X) \rightarrow \mathcal{E}(Y)$ is an order-isomorphism. Then, since $\phi(\mathcal{E}_K(X))$ satisfies the conditions of Theorem 3.6, by maximality of $\mathcal{E}_K(Y)$, we have $\phi(\mathcal{E}_K(X)) \subseteq \mathcal{E}_K(Y)$, and therefore by symmetry $\phi(\mathcal{E}_K(X)) = \mathcal{E}_K(Y)$. Thus $\mathcal{E}_K(X)$ and $\mathcal{E}_K(Y)$ are order-isomorphic. Note that in proof of Theorem 4.1, using its notations, we could define a homeomorphism $f : \omega\sigma X \setminus X \rightarrow \omega\sigma Y \setminus Y$ such that $f(U) = G(U)$, for any $U \in \mathcal{B}(\omega\sigma X \setminus X)$. Therefore since for any countable $J \subseteq I$, $\Omega' \notin g(Q_J) = G(Q_J) = f(Q_J)$, we have $f(\Omega) = \Omega'$ and thus $f|_{\sigma X \setminus X} : \sigma X \setminus X \rightarrow \sigma Y \setminus Y$ is a homeomorphism. In other words, having $\mathcal{E}(X)$ and $\mathcal{E}(Y)$ order-isomorphic implies that $\sigma X \setminus X$ and $\sigma Y \setminus Y$ are homeomorphic. We don't know if the converse also holds. More precisely

Question 7.1. *Let X be a locally compact non-separable metrizable space. Is there a subspace of X^* (in particular X^* itself) whose topology determines and is determined by the order structure of the set $\mathcal{E}(X)$?*

Question 7.2. *Let X and Y be zero-dimensional locally compact non-separable metrizable spaces. Is every order-isomorphism $\psi : \mathcal{E}_K(X) \rightarrow \mathcal{E}_K(Y)$ extendable to one from $\mathcal{E}(X)$ onto $\mathcal{E}(Y)$? Are at least $\mathcal{E}(X)$ and $\mathcal{E}(Y)$ order-isomorphic?*

It turns out that the above two questions are related in the following way. Suppose that for every zero-dimensional locally compact non-separable metrizable spaces X and Y , any order-isomorphism $\phi : \mathcal{E}(X) \rightarrow \mathcal{E}(Y)$, induces a homeomorphism $f : X^* \rightarrow Y^*$, in such a way that for every $T \in \lambda_X(\mathcal{E}(X))$, $f(T) = \lambda_Y \phi \lambda_X^{-1}(T)$. Let $X = D(\aleph_1)$ and $Y = \bigoplus_{i < \omega_1} Y_i$, where for each $i < \omega_1$, Y_i is the one-point compactification of ω . Then by Theorem 4.4, under [CH], $\mathcal{E}_K(X)$ and $\mathcal{E}_K(Y)$ are order-isomorphic. Suppose that $\mathcal{E}(X)$ and $\mathcal{E}(Y)$ are order-isomorphic, and let ϕ and f be as defined above. Then as the proof of Theorem 2.1 shows, there exists a non-empty zero-set $Z \in \mathcal{Z}(Y^*) \setminus \lambda_Y(\mathcal{E}(Y))$ such that $\text{int}_{\sigma Y}(Z \setminus \sigma Y) = \emptyset$. By

Lemma 5.8, we have $f(X^* \setminus \sigma X) = Y^* \setminus \sigma Y$ and therefore $\text{int}_{\sigma X}(f^{-1}(Z) \setminus \sigma X) = \emptyset$. By Theorem 6.8 of [9], this implies that $f^{-1}(Z) \in \lambda_X(\mathcal{E}(X))$, which clearly contradicts our assumptions. Therefore, in some ways the answer to the above two questions cannot both be positive.

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